1. (20) Consider a wide sense stationary random sequence $X[n]$ with mean function $\mu_X$, a constant, and correlation function $R_{XX}[m]$. Form a random process as

$$X(t) = \sum_{n=-\infty}^{\infty} X[n] \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T} \quad -\infty < t < \infty$$

In what follows, we assume the infinite sums converge, and so, do not worry about stochastic convergence issues.

(a) Find $\mu_X(t)$ in terms of $\mu_X$. Simplify your answer as much as possible.

(b) Find $R_{XX}(t_1, t_2)$ in terms of $R_{XX}[m]$. Is $X(t)$ WSS?

*Hint:* The sampling theorem from linear systems theory states that any bandlimited deterministic function $g(t)$ can be recovered exactly from its evenly spaced samples, i.e.,

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T},$$

when the radian bandwidth of the function $g(t)$ is $\pi/T$ or less.

2. (15) Let the random process $Y(t)$ be given as

$$Y(t) = X(t) + 0.3 \frac{dX(t)}{dt},$$

where $X(t)$ is a random process with mean function $\mu_X(t) = 5t$, and covariance function

$$C_{XX}(t_1, t_2) = \frac{\sigma^2}{1 + \alpha(t_1 - t_2)^2}, \quad \alpha > 0$$

(a) Find the mean function $\mu_Y(t)$.

(b) Find the covariance function $C_{YY}(t_1, t_2)$.

(c) Is the random process $Y(t)$ wide sense stationary? Why?
3. (20 points) A wide sense stationary and zero-mean random process \( Y(t) \) has sample functions consisting of successive rectangular pulses of random amplitude and duration as shown below. The pdf for the pulse width is

\[
f_W(w) = \begin{cases} 
\lambda e^{-\lambda w}, & w \geq 0, \\
0, & w < 0,
\end{cases}
\]

with \( \lambda > 0 \). The amplitude of each pulse is a random variable \( X \) (independent of \( W \)) with mean 0 and variance \( \sigma_X^2 \). Successive amplitudes and pulse widths are independent.

(a) Find the autocorrelation function \( R_{YY}(\tau) = E[Y(t + \tau)Y(t)] \).

(b) Find the corresponding power spectral density \( S_{YY}(\omega) \).

[Hint: First find the conditional autocorrelation function \( E[Y(t + \tau)Y(t)|W = w] \), where \( t \) is assumed to be at the start of a pulse (per WSS assumption for \( Y(t) \)).]

4. (20) We consider here the idea of using white noise as an approximation to a smoother process which is input to a lowpass filter. The output process from the filter is then investigated to determine the error resulting from the white noise approximation. Let the stationary random process \( X(t) \) have zero-mean and autocovariance function

\[
C_{XX}(\tau) = \frac{1}{2\tau_0} \exp(-|\tau|/\tau_0)
\]

which can be written as \( h(\tau) * h(-\tau) \) with \( h(\tau) = \frac{1}{\tau_0} e^{-\tau/\tau_0} u(\tau) \).

(a) Let \( X(t) \) be input to a lowpass filter with transfer function

\[
G(\omega) = \begin{cases} 
1, & |\omega| < \omega_0 \\
0, & \text{else}
\end{cases}
\]

Denote the filter output by \( Y(t) \) and find the output power spectral density \( S_{YY}(\omega) \).

(b) Alternatively, we may excite the system above directly with a standard white noise process \( W(t) \), with mean zero and \( C_{WW}(\tau) = \delta(\tau) \). Call the output \( V(t) \) and find the output power spectral density \( S_{VV}(\omega) \).

(c) Show that for \( |\omega_0\tau_0| \ll 1 \), \( S_{YY} \approx S_{VV} \), and find an upper bound on the power error

\[
|R_{VV}(0) - R_{YY}(0)|.
\]

[Hint: Upper bound \( \frac{(\omega_0\tau_0)^2}{1+(\omega_0\tau_0)^2} \) with \( \frac{(\omega_0\tau_0)^2}{1+(\omega_0\tau_0)^2} \).]
5. (25) Let $N(t)$ be a Poisson process, and suppose we form a *phase modulated* random process

$$Y(t) = a \cos(2\pi ft + \pi N(t))$$

(a) Plot a sample function of $Y(t)$ corresponding to a typical sample function of $N(t)$.
(b) Find the joint density function of $Y(t_1)$ and $Y(t_2)$. *Hint:* Use the independent increments property of $N(t)$.
(c) Find the mean and autocorrelation functions of $Y(t)$.
(d) Is $Y(t)$ strictly stationary or wide sense stationary? Explain.
(e) A random process $Z(t)$ is called wide sense *cyclostationary* if there is some $T > 0$ such that its mean

$$\mu_Z(t) = \mu_Z(t + kT)$$

for all $t$ and $k$, and, its autocorrelation

$$R_{ZZ}(t_1, t_2) = R_{ZZ}(t_1 + kT, t_2 + kT)$$

for all $t_1, t_2$, and $k$. Is $Y(t)$ wide sense cyclostationary, or *asymptotically* wide sense cyclostationary (i.e., as $t_1 \to \infty$)? Explain.