California Institute of Technology
Department of Electrical Engineering

EE 162
Random Processes for Communication and Signal Processing

Fall 2004

http://ee162.caltech.edu

Final Exam 12-02-2004

This is an open book and notes, but closed friends, family, and neighbors exam. You are permitted to use online resources, as long as it does not involve requests for help of any type, through email, chat rooms, instant messaging, etc, from other actual human beings (except me, the instructor). You do not need to sit through the exam at once, and can take breaks, go out, eat, sleep, watch Saturday Night Live, ... as long as the total amount of time you spend on solving the problems is less than 6-7 hours.

All the work in this exam is mine alone. I have neither received assistance from another person or group, nor have I given assistance to another person.

Name: ________________________ **Signature/Date:** ________________________

1. (20) Consider a wide sense stationary random sequence $X[n]$ with mean function μ_X, a constant, and correlation function $R_{XX}[m]$. Form a random process as

$$X(t) = \sum_{n=-\infty}^{\infty} X[n] \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T} \qquad -\infty < t < \infty$$

   In what follows, we assume the infinite sums converge, and so, do not worry about stochastic convergence issues.

   (a) Find $\mu_X(t)$ in terms of μ_X. Simplify your answer as much as possible.

   (b) Find $R_{XX}(t_1, t_2)$ in terms of $R_{XX}[m]$. Is $X(t)$ WSS?

   Hint: The sampling theorem from linear systems theory states that any bandlimited deterministic function $g(t)$ can be recovered exactly from its evenly spaced samples, i.e.,

$$g(t) = \sum_{n=-\infty}^{\infty} g(nT) \frac{\sin \pi(t - nT)/T}{\pi(t - nT)/T},$$

   when the radian bandwidth of the function $g(t)$ is π/T or less.

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2. (15) Let the random process $Y(t)$ be given as

$$Y(t) = X(t) + 0.3 \frac{dX(t)}{dt},$$

   where $X(t)$ is a random process with mean function $\mu_X(t) = 5t$, and covariance function

$$C_{XX}(t_1, t_2) = \frac{\sigma^2}{1 + \alpha(t_1 - t_2)^2}, \qquad \alpha > 0$$

   (a) Find the mean function $\mu_Y(t)$.

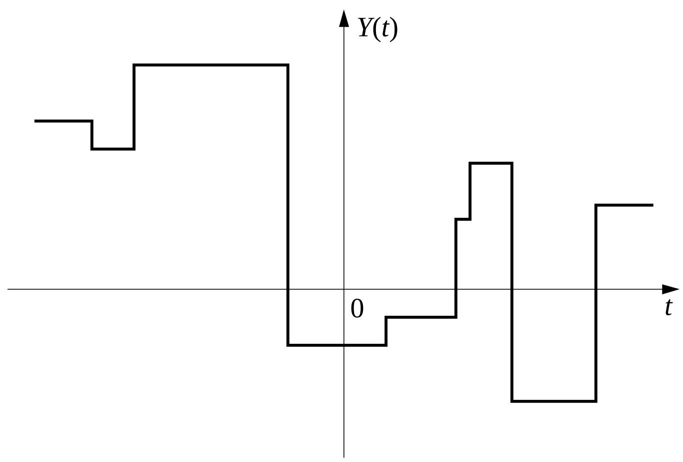
   (b) Find the covariance function $C_{YY}(t_1, t_2)$.

   (c) Is the random process $Y(t)$ wide sense stationary? Why?

3. (20 points) A wide sense stationary and zero-mean random process $Y(t)$ has sample functions consisting of successive rectangular pulses of random amplitude and duration as shown below. The pdf for the pulse width is

$$f_W(w) = \begin{cases} \lambda e^{-\lambda w}, & w \geq 0, \\ 0, & w < 0, \end{cases}$$

with $\lambda > 0$. The amplitude of each pulse is a random variable X (independent of W) with mean 0 and variance σ_X^2. Successive amplitudes and pulse widths are independent.



(a) Find the autocorrelation function $R_{YY}(\tau) = E[Y(t+\tau)Y(t)]$.

(b) Find the corresponding power spectral density $S_{YY}(\omega)$.

[*Hint:* First find the *conditional* autocorrelation function $E[Y(t+\tau)Y(t)|W = w]$, where t is assumed to be at the start of a pulse (per WSS assumption for $Y(t)$)].

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4. (20) We consider here the idea of using white noise as an approximation to a smoother process which is input to a lowpass filter. The output process from the filter is then investigated to determine the error resulting from the white noise approximation. Let the stationary random process $X(t)$ have zero-mean and autocovariance function

$$C_{XX}(\tau) = \frac{1}{2\tau_0} \exp(-|\tau|/\tau_0)$$

which can be written as $h(\tau) * h(-\tau)$ with $h(\tau) = \frac{1}{\tau_0} e^{-\tau/\tau_0} u(\tau)$.

(a) Let $X(t)$ be input to a lowpass filter with transfer function

$$G(\omega) = \begin{cases} 1, & |\omega| < \omega_0 \\ 0, & \text{else} \end{cases}$$

Denote the filter output by $Y(t)$ and find the output power spectral density $S_{YY}(\omega)$.

(b) Alternatively, we may exite the system above directly with a standard white noise process $W(t)$, with mean zero and $C_{WW}(\tau) = \delta(\tau)$. Call the output $V(t)$ and find the output power spectral density $S_{VV}(\omega)$.

(c) Show that for $|\omega_0 \tau_0| \ll 1$, $S_{YY} \approx S_{VV}$, and find an upper bound on the *power error*

$$|R_{VV}(0) - R_{YY}(0)|.$$

Hint: Upper bound $\frac{(\omega\tau_0)^2}{1+(\omega\tau_0)^2}$ with $\frac{(\omega_0\tau_0)^2}{1+(\omega_0\tau_0)^2}$.

5. (25) Let $N(t)$ be a Poisson process, and suppose we form a *phase modulated* random process

$$Y(t) = a\cos(2\pi ft + \pi N(t))$$

(a) Plot a sample function of $Y(t)$ corresponding to a typical sample function of $N(t)$.

(b) Find the joint density function of $Y(t_1)$ and $Y(t_2)$. *Hint:* Use the independent increments property of $N(t)$.

(c) Find the mean and autocorrelation functions of $Y(t)$.

(d) Is $Y(t)$ strictly stationary or wide sense stationary? Explain.

(e) A random process $Z(t)$ is called wide sense *cyclostationary* if there is some $T > 0$ such that its mean

$$\mu_Z(t) = \mu_Z(t + kT)$$

for all t and k, and, its autocorrelation

$$R_{ZZ}(t_1, t_2) = R_{ZZ}(t_1 + kT, t_2 + kT)$$

for all t_1, t_2, and k. Is $Y(t)$ wide sense cyclostationary, or *aysmptotically* wide sense cyclostationary (i.e., as $t_1 \to \infty$)? Explain.