California Institute of Technology
Department of Electrical Engineering

EE 162
Random Processes for Communication and Signal Processing

Fall 2004

http://ee162.caltech.edu

Midterm Exam 10-28-2004

This is an open book and notes, but closed friends, family, and neighbors exam. You are permitted to use online resources, as long as it does not involve requests for help of any type, through email, chat rooms, instant messaging, etc, from other actual human beings (except me, the instructor). You do not need to sit through the exam at once, and can take breaks, go out, eat, sleep, watch Saturday Night Live, ... as long as the total amount of time you spend on solving the problems is less than 6-7 hours.

All the work in this exam is mine alone. I have neither received assistance from another person or group, nor have I given assistance to another person.

Name: ______________________________ **Signature/Date:** ______________________________

1. (15) Let X be a uniform random variable on $[0, 2]$. Compute the pdf of $Y = g(X)$ (see Fig. P1 below).

2. (20) One of two constant values, a or b, is transmitted across a noisy communication channel. At the receiver the signal is passed through an amplifier (unity gain, for our purposes here) whose output, y, is used to determine which value was sent. Specifically, when an a is transmitted, event H_a, we have $Y = a + N$, and when a b is transmitted, event H_b, we have $Y = b + N$, where N is a zero mean Gaussian random variable with variance σ^2. Given $P\{H_a\} = \rho$ and $P\{H_b\} = 1 - \rho$:
   (a) Write down, by observation, the conditional density functions $f_Y(y|H_a)$ and $f_Y(y|H_b)$.
   (b) Find $P\{H_a|Y = y\}$ and $P\{H_b|Y = y\}$.
   (c) To minimize the probability of error for what region of y should we decide an a was sent? For what region of y should we decide a b was sent? Give your answers for both cases of $a > b$ and $a < b$. (*Hint:* use your answers from part b).
   (d) For the case when $b = -a$ and $\rho = \frac{1}{2}$, calculate the minimum attainable probability of error (in terms of the complementary error function).
   (e) Now let's say the channel has *fading* in addition to additive noise, so that we model the received signal as either $Y = Za + N$, or $Y = Zb + N$, where Z is an exponential random variable, i.e., $f_Z(z) = \exp(-z)u(z)$. Z is independent of N. For the case when an a is received ($Y = Za + N$) determine $f_Y(y|Z > z_o)$. For simplicity assume that $a = \sigma = z_0 = 1$.

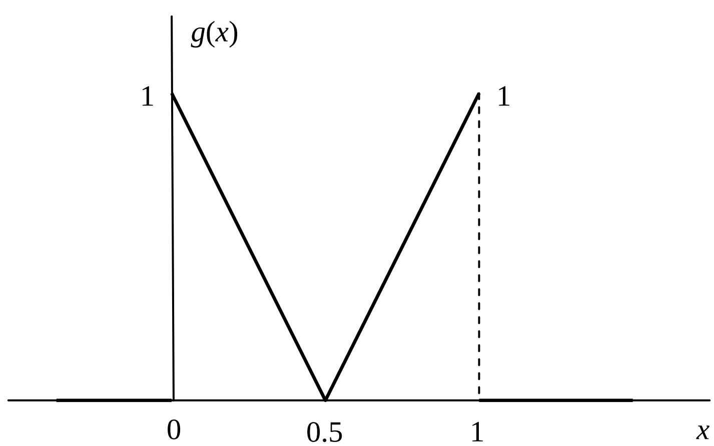

Figure P1

3. (25) Problem 4.27 in your text (page 142).

4. (20) Consider the function

$$f(x,y) = f_X(x) f_Y(y)[1 + \rho\{2F_X(x) - 1\}\{2F_Y(y) - 1\}] \qquad |\rho| < 1$$

where $f_X(x)$ and $f_Y(y)$ are two pdfs with respective distribution functions $F_X(x)$ and $F_Y(y)$.

(a) Is $f(x,y) \geq 0$ for all x, y? Explain your reasoning. Now, compute

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

Can $f(x,y)$ represent a joint pdf of two random variables X and Y, or not?

(b) Determine

$$\int_{-\infty}^{\infty} f(x,y)\, dy \quad \text{and} \quad \int_{-\infty}^{\infty} f(x,y)\, dx$$

(c) Now assume that $f_X(x)$ and $f_Y(y)$ are normally distributed with respective means and variances (η_1, σ_1^2) and (η_2, σ_2^2).

Which statement below is true? Explain.

c1. $f(x,y)$ a jointly normal pdf with normal marginals.
c2. $f(x,y)$ a joint pdf with normal marginals, that is not jointly normal.
c3. $f(x,y)$ a jointly normal pdf without normal marginals.
c4. $f(x,y)$ a joint pdf without normal marginals, that is not jointly normal.

5. (20) Two countries A and B are engaged in a last ditch diplomatic effort to resolve their differences before war breaks out. As such, they engage in a series of face to face talks. Each country's goal during a meeting session is to exclusively win a concession from the other, without giving in on any issue (i.e. either country A *or* country B wins a concession per session, but not both at the same time). Assume that country A is always at a disadvantage in the talks, so that the probability of it winning a concession in each talk is $p < 1/2$. However, country A gets to choose in advance the total number of meeting sessions. To "win" the series of talks, one must win concessions from the other side in more than half of the sessions. If the total number of sessions is to be even, how many sessions should A choose in advance? Find an expression for the optimum number of sessions, $2n$, in terms of p, and compute it for $p = 0.47$ and $p \approx 0$. What do you observe for this last case?

Hint: We must have

$$P_{2n-2} \leq P_{2n} \geq P_{2n+2}$$

where for instance, P_{2n+2} denotes the probability that A wins in $2n + 2$ sessions.