12. Stochastic Processes

Introduction

Let ξ denote the random outcome of an experiment. To every such outcome suppose a waveform $X(t,\xi)$ is assigned. The collection of such waveforms form a stochastic process. The set of $\{\xi_k\}$ and the time index t can be continuous or discrete (countably infinite or finite) as well.

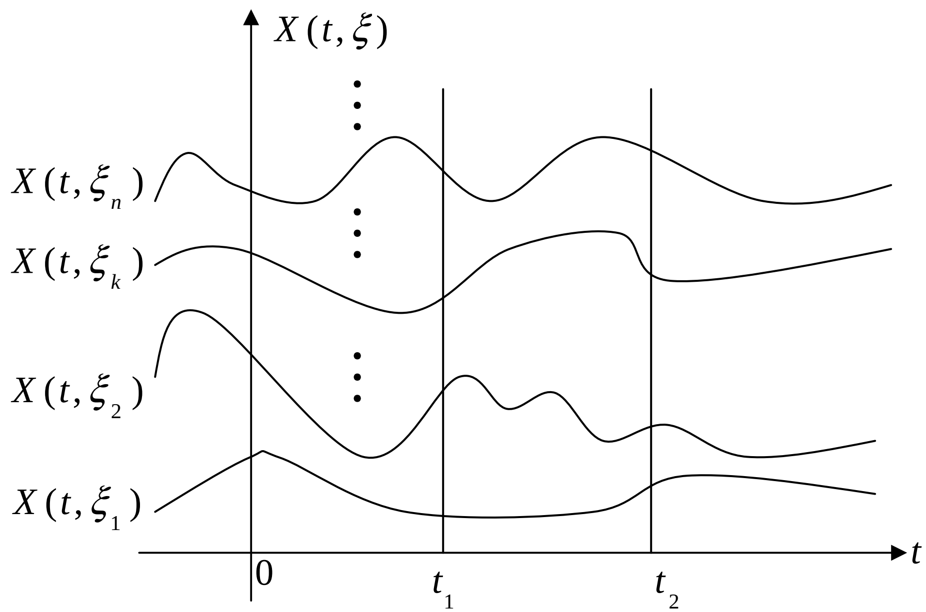

Fig. 12.1

For fixed $\xi_i \in S$ (the set of all experimental outcomes), $X(t,\xi)$ is a specific time function. For fixed t,

$$X_1 = X(t_1, \xi_i)$$

is a random variable. The ensemble of all such realizations $X(t,\xi)$ over time represents the stochastic process $X(t)$. (see Fig 12.1). For example

$$X(t) = a\cos(\omega_0 t + \varphi),$$

where φ is a uniformly distributed random variable in $(0, 2\pi)$, represents a stochastic process. Stochastic processes are everywhere: Brownian motion, stock market fluctuations, various queuing systems all represent stochastic phenomena.

If $X(t)$ is a stochastic process, then for fixed t, $X(t)$ represents a random variable. Its distribution function is given by

$$F_X(x,t) = P\{X(t) \le x\} \tag{12-1}$$

Notice that $F_X(x,t)$ depends on t, since for a different t, we obtain a different random variable. Further

$$f_X(x,t) \triangleq \frac{dF_X(x,t)}{dx} \tag{12-2}$$

represents the first-order probability density function of the process $X(t)$.

For $t = t_1$ and $t = t_2$, $X(t)$ represents two different random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ respectively. Their joint distribution is given by

$$F_X(x_1, x_2, t_1, t_2) = P\{X(t_1) \le x_1, X(t_2) \le x_2\} \qquad (12\text{-}3)$$

and

$$f_X(x_1, x_2, t_1, t_2) \triangleq \frac{\partial^2 F_X(x_1, x_2, t_1, t_2)}{\partial x_1 \, \partial x_2} \qquad (12\text{-}4)$$

represents the second-order density function of the process $X(t)$. Similarly $f_X(x_1, x_2, \cdots x_n,\ t_1, t_2 \cdots, t_n)$ represents the nth order density function of the process $X(t)$. Complete specification of the stochastic process $X(t)$ requires the knowledge of $f_X(x_1, x_2, \cdots x_n,\ t_1, t_2 \cdots, t_n)$ for all $t_i, \quad i = 1, 2, \cdots, n$ and for all n. (an almost impossible task in reality).

Mean of a Stochastic Process:

$$\mu(t) \triangleq E\{X(t)\} = \int_{-\infty}^{+\infty} x f_X(x,t)dx \qquad (12\text{-}5)$$

represents the mean value of a process $X(t)$. In general, the mean of a process can depend on the time index t.

Autocorrelation function of a process $X(t)$ is defined as

$$R_{XX}(t_1,t_2) \triangleq E\{X(t_1)X^*(t_2)\} = \int\int x_1 x_2^* f_X(x_1,x_2,t_1,t_2)dx_1dx_2 \qquad (12\text{-}6)$$

and it represents the interrelationship between the random variables $X_1 = X(t_1)$ and $X_2 = X(t_2)$ generated from the process $X(t)$.

Properties:

1. $R_{XX}(t_1,t_2) = R_{XX}^*(t_2,t_1) = [E\{X(t_2)X^*(t_1)\}]^* \qquad (12\text{-}7)$

2. $R_{XX}(t,t) = E\{|X(t)|^2\} > 0.$ (Average instantaneous power)

3. $R_{XX}(t_1, t_2)$ represents a nonnegative definite function, i.e., for *any* set of constants $\{a_i\}_{i=1}^n$

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i, t_j) \geq 0. \qquad (12\text{-}8)$$

Eq. (12-8) follows by noticing that $E\{|Y|^2\} \geq 0$ for $Y = \sum_{i=1}^{n} a_i X(t_i).$ The function

$$C_{XX}(t_1, t_2) = R_{XX}(t_1, t_2) - \mu_X(t_1)\mu_X^*(t_2) \qquad (12\text{-}9)$$

represents the **autocovariance** function of the process $X(t)$.

Example 12.1

Let

$$z = \int_{-T}^{T} X(t)dt.$$

Then

$$E[|z|^2] = \int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}dt_1 dt_2$$

$$= \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1, t_2)dt_1 dt_2 \qquad (12\text{-}10)$$

Example 12.2

$$X(t) = a\cos(\omega_0 t + \varphi), \qquad \varphi \sim U(0, 2\pi). \tag{12-11}$$

This gives

$$\mu_X(t) = E\{X(t)\} = aE\{\cos(\omega_0 t + \varphi)\}$$
$$= a\cos\omega_0 t\, E\{\cos\varphi\} - a\sin\omega_0 t\, E\{\sin\varphi\} = 0, \tag{12-12}$$

since $E\{\cos\varphi\} = \frac{1}{2\pi}\int_0^{2\pi}\cos\varphi\, d\varphi = 0 = E\{\sin\varphi\}.$

Similarly

$$R_{XX}(t_1, t_2) = a^2 E\{\cos(\omega_0 t_1 + \varphi)\cos(\omega_0 t_2 + \varphi)\}$$
$$= \frac{a^2}{2} E\{\cos\omega_0(t_1 - t_2) + \cos(\omega_0(t_1 + t_2) + 2\varphi)\}$$
$$= \frac{a^2}{2}\cos\omega_0(t_1 - t_2). \tag{12-13}$$

Stationary Stochastic Processes

Stationary processes exhibit statistical properties that are invariant to shift in the time index. Thus, for example, second-order stationarity implies that the statistical properties of the pairs $\{X(t_1), X(t_2)\}$ and $\{X(t_1+c), X(t_2+c)\}$ are the same for *any* c. Similarly first-order stationarity implies that the statistical properties of $X(t_i)$ and $X(t_i+c)$ are the same for any c.

In strict terms, the statistical properties are governed by the joint probability density function. Hence a process is n^{th}-order **Strict-Sense Stationary (S.S.S)** if

$$f_X(x_1, x_2, \cdots x_n, \ t_1, t_2 \cdots, t_n) \equiv f_X(x_1, x_2, \cdots x_n, \ t_1 + c, t_2 + c \cdots, t_n + c) \tag{12-12}$$

for *any* c, where the left side represents the joint density function of the random variables $X_1 = X(t_1), \ X_2 = X(t_2), \ \cdots, \ X_n = X(t_n)$ and the right side corresponds to the joint density function of the random variables $X_1' = X(t_1 + c), \ X_2' = X(t_2 + c), \ \cdots, \ X_n' = X(t_n + c)$.

A process $X(t)$ is said to be **strict-sense stationary** if (12-12) is true for all $t_i, \ \ i = 1, 2, \cdots, n, \ \ n = 1, 2, \cdots$ and *any* c.

For a **first-order strict sense stationary process**, from (12-12) we have

$$f_X(x,t) \equiv f_X(x,t+c) \tag{12-15}$$

for any c. In particular $c = -t$ gives

$$f_X(x,t) = f_X(x) \tag{12-16}$$

i.e., the first-order density of $X(t)$ is independent of t. In that case

$$E[X(t)] = \int_{-\infty}^{+\infty} x\, f(x)dx = \mu, \ \ a \ constant. \tag{12-17}$$

Similarly, for a **second-order strict-sense stationary process** we have from (12-12)

$$f_X(x_1, x_2, \ t_1, t_2) \equiv f_X(x_1, x_2, \ t_1 + c, t_2 + c)$$

for any c. For $c = -t_2$ we get

$$f_X(x_1, x_2, \ t_1, t_2) \equiv f_X(x_1, x_2, \ t_1 - t_2) \tag{12-18}$$

i.e., the second order density function of a strict sense stationary process depends only on the difference of the time indices $t_1 - t_2 = \tau$. In that case the autocorrelation function is given by

$$\begin{aligned} R_{XX}(t_1, t_2) &\triangleq E\{X(t_1)X^*(t_2)\} \\ &= \int\int x_1 x_2^* \, f_X(x_1, x_2, \tau = t_1 - t_2) dx_1 dx_2 \\ &= R_{XX}(t_1 - t_2) \triangleq R_{XX}(\tau) = R_{XX}^*(-\tau), \end{aligned} \qquad (12\text{-}19)$$

i.e., the autocorrelation function of a second order strict-sense stationary process depends only on the difference of the time indices $\tau = t_1 - t_2$.

Notice that (12-17) and (12-19) are consequences of the stochastic process being first and second-order strict sense stationary.

On the other hand, the basic conditions for the first and second order stationarity – Eqs. (12-16) and (12-18) – are usually difficult to verify. In that case, we often resort to a looser definition of stationarity, known as **Wide-Sense Stationarity (W.S.S),** by making use of

(12-17) and (12-19) as the necessary conditions. Thus, a process $X(t)$ is said to be **Wide-Sense Stationary** if

(i) $E\{X(t)\} = \mu$ (12-20)

and

(ii) $E\{X(t_1)X^*(t_2)\} = R_{XX}(t_1 - t_2),$ (12-21)

i.e., for wide-sense stationary processes, the mean is a constant and the autocorrelation function depends only on the difference between the time indices. Notice that (12-20)-(12-21) does not say anything about the nature of the probability density functions, and instead deal with the average behavior of the process. Since (12-20)-(12-21) follow from (12-16) and (12-18), strict-sense stationarity always implies wide-sense stationarity. However, the converse is *not true* in general, the only exception being the Gaussian process.

This follows, since if $X(t)$ is a Gaussian process, then by definition $X_1 = X(t_1), X_2 = X(t_2), \cdots, X_n = X(t_n)$ are jointly Gaussian random variables for any $t_1, t_2 \cdots, t_n$ whose joint characteristic function is given by

$$\phi_{\underline{X}}(\omega_1,\omega_2,\cdots,\omega_n)=e^{\,j\sum_{k=1}^{n}\mu(t_k)\omega_k-\sum_{l,k}^{n}\sum C_{xx}(t_i,t_k)\omega_i\omega_k/2} \tag{12-22}$$

where $C_{XX}(t_i,t_k)$ is as defined on (12-9). If $X(t)$ is wide-sense stationary, then using (12-20)-(12-21) in (12-22) we get

$$\phi_{\underline{X}}(\omega_1,\omega_2,\cdots,\omega_n)=e^{\,j\sum_{k=1}^{n}\mu\omega_k-\frac{1}{2}\sum_{1=1}^{n}\sum_{k=1}^{n}C_{xx}(t_i-t_k)\omega_i\omega_k} \tag{12-23}$$

and hence if the set of time indices are shifted by a constant c to generate a new set of jointly Gaussian random variables $X_1'=X(t_1+c)$, $X_2'=X(t_2+c),\cdots,X_n'=X(t_n+c)$ then their joint characteristic function is identical to (12-23). Thus the set of random variables $\{X_i\}_{i=1}^{n}$ and $\{X_i'\}_{i=1}^{n}$ have the same joint probability distribution for all n and all c, establishing the strict sense stationarity of Gaussian processes from its wide-sense stationarity.

To summarize if $X(t)$ is a Gaussian process, then

wide-sense stationarity (w.s.s) $\Rightarrow$ strict-sense stationarity (s.s.s).

Notice that since the joint p.d.f of Gaussian random variables depends only on their second order statistics, which is also the basis for wide sense stationarity, we obtain strict sense stationarity as well. From (12-12)-(12-13), (refer to Example 12.2), the process $X(t) = a\cos(\omega_0 t + \varphi)$, in (12-11) is wide-sense stationary, but not strict-sense stationary.

Similarly if $X(t)$ is a zero mean wide sense stationary process in Example 12.1, then σ_z^2 in (12-10) reduces to

$$\sigma_z^2 = E\{|z|^2\} = \int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)\,dt_1 dt_2 .$$

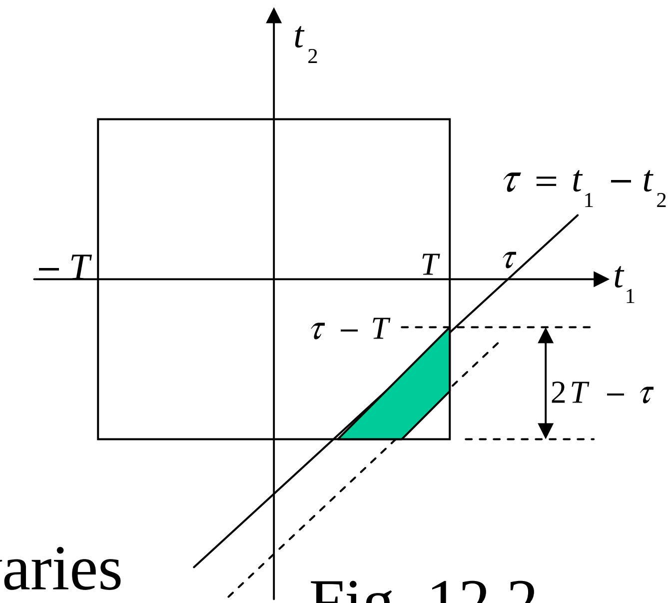

Fig. 12.2

As t_1, t_2 varies from $-T$ to $+T$, $\tau = t_1 - t_2$ varies from $-2T$ to $+2T$. Moreover $R_{XX}(\tau)$ is a constant over the shaded region in Fig 12.2, whose area is given by $(\tau > 0)$

$$\frac{1}{2}(2T-\tau)^2 - \frac{1}{2}(2T-\tau-d\tau)^2 = (2T-\tau)d\tau$$

and hence the above integral reduces to

$$\sigma_z^2 = \int_{-2t}^{2T} R_{XX}(\tau)(2T - |\tau|)d\tau = \frac{1}{2T}\int_{-2t}^{2T} R_{XX}(\tau)\left(1 - \frac{|\tau|}{2T}\right)d\tau. \tag{12-24}$$

Systems with Stochastic Inputs

A deterministic system[1] transforms each input waveform $X(t, \xi_i)$ into an output waveform $Y(t, \xi_i) = T[X(t, \xi_i)]$ by operating only on the time variable *t*. Thus a set of realizations at the input corresponding to a process *X*(*t*) generates a new set of realizations $\{Y(t, \xi)\}$ at the output associated with a new process *Y*(*t*).

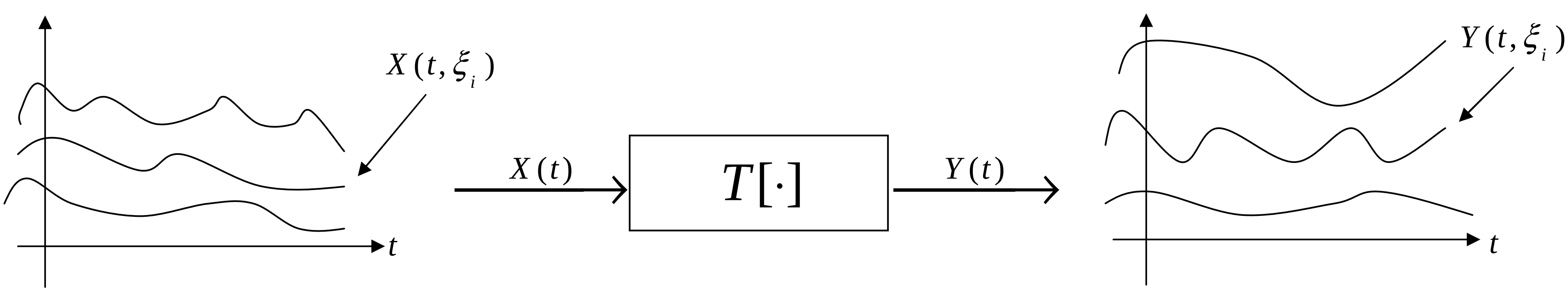

Fig. 12.3

Our goal is to study the output process statistics in terms of the input process statistics and the system function.

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[1]A stochastic system on the other hand operates on both the variables *t* and ξ.

Deterministic Systems

Memoryless Systems

$$Y(t) = g[X(t)]$$

Systems with Memory

Time-varying systems

Time-Invariant systems

Linear systems

$$Y(t) = L[X(t)]$$

Fig. 12.3

Linear-Time Invariant (LTI) systems

$X(t) \longrightarrow$ $h(t)$ $\longrightarrow$

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau$$

$$= \int_{-\infty}^{+\infty} h(\tau)X(t-\tau)d\tau.$$

LTI system

Memoryless Systems:

The output $Y(t)$ in this case depends only on the present value of the input $X(t)$. i.e.,

$$Y(t) = g\{X(t)\} \tag{12-25}$$

Strict-sense stationary input → Memoryless system → Strict-sense stationary output.

(see (9-76), Text for a proof.)

Wide-sense stationary input → Memoryless system → Need *not* be stationary in any sense.

$X(t)$ stationary Gaussian with $R_{XX}(\tau)$ → Memoryless system → $Y(t)$ stationary,but *not* Gaussian with $R_{XY}(\tau) = \eta R_{XX}(\tau)$. (see (12-26)).

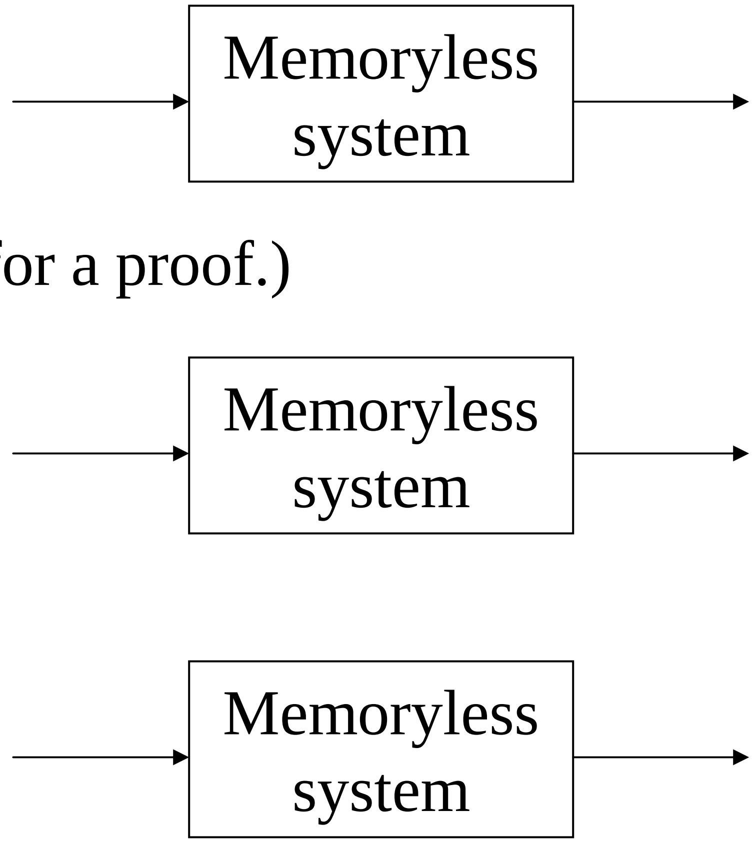


Fig. 12.4

Theorem: If $X(t)$ is a zero mean stationary Gaussian process, and $Y(t) = g[X(t)]$, where $g(\cdot)$ represents a nonlinear memoryless device, then

$$R_{XY}(\tau) = \eta R_{XX}(\tau), \qquad \eta = E\{g'(X)\}. \tag{12-26}$$

Proof:

$$\begin{aligned} R_{XY}(\tau) &= E\{X(t)Y(t-\tau)\} = E[X(t)g\{X(t-\tau)\}] \\ &= \int\int x_1 g(x_2) f_{X_1X_2}(x_1, x_2) dx_1 dx_2 \end{aligned} \tag{12-27}$$

where $X_1 = X(t), \quad X_2 = X(t-\tau)$ are jointly Gaussian random variables, and hence

$$f_{X_1X_2}(x_1, x_2) = \frac{1}{2\pi\sqrt{|A|}} e^{-\underline{x}^* A^{-1} \underline{x}/2}$$

$$\underline{X} = (X_1, X_2)^T, \qquad \underline{x} = (x_1, x_2)^T$$

$$A = E\{\underline{X}\,\underline{X}^*\} = \begin{pmatrix} R_{XX}(0) & R_{XX}(\tau) \\ R_{XX}(\tau) & R_{XX}(0) \end{pmatrix} \triangleq LL^*$$

where L is an upper triangular factor matrix with positive diagonal entries. i.e.,

$$L = \begin{pmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{pmatrix}.$$

Consider the transformation

$$\underline{Z} \triangleq L^{-1}\underline{X} = (Z_1,\ Z_2)^T, \qquad \underline{z} \triangleq L^{-1}\underline{x} = (z_1,\ z_2)^T$$

so that

$$E\{\underline{Z}\,\underline{Z}^*\} = L^{-1}E\{\underline{X}\,X^*\}L^{*^{-1}} = L^{-1}AL^{*^{-1}} = I$$

and hence Z_1, Z_2 are zero mean independent Gaussian random variables. Also

$$\underline{x} = L\underline{z} \;\Rightarrow\; x_1 = l_{11}z_1 + l_{12}z_2, \quad x_2 = l_{22}z_2$$

and hence

$$\underline{x}^* A^{-1}\underline{x} = \underline{z}^* L^* A^{-1} L\underline{z} = \underline{z}^*\underline{z} = z_1^2 + z_2^2.$$

The Jacobaian of the transformation is given by

$$|J| = |L^{-1}| = |A|^{-1/2}.$$

Hence substituting these into (12-27), we obtain

$$R_{XY}(\tau) = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}(l_{11}z_1 + l_{12}z_2)g(l_{22}z_2)\tfrac{1}{|J|}\cdot\tfrac{1}{2\pi|A|^{1/2}}e^{-z_1^2/2}e^{-z_2^2/2}$$

$$= l_{11}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z_1 g(l_{22}z_2) f_{z_1}(z_1) f_{z_2}(z_2)dz_1 dz_2$$

$$+ l_{12}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} z_2 g(l_{22}z_2) f_{z_1}(z_1) f_{z_2}(z_2)dz_1 dz_2$$

$$= l_{11}\underbrace{\int_{-\infty}^{+\infty} z_1 f_{z_1}(z_1)dz_1}_{0}\int_{-\infty}^{+\infty} g(l_{22}z_2) f_{z_2}(z_2)dz_2$$

$$+ l_{12}\int_{-\infty}^{+\infty} z_2 g(l_{22}z_2)\underbrace{f_{z_2}(z_2)}_{\frac{1}{\sqrt{2\pi}}e^{-z_2^2/2}}dz_2$$

$$= \frac{l_{12}}{l_{22}^2}\int_{-\infty}^{+\infty} ug(u)\tfrac{1}{\sqrt{2\pi}}e^{-u^2/2l_{22}^2}du,$$

where $u = l_{22}z_2$. This gives

$$R_{XY}(\tau) = l_{12} l_{22} \int_{-\infty}^{+\infty} g(u) \underbrace{\frac{u}{l_{22}^2} \overbrace{\frac{1}{\sqrt{2\pi\, l_{22}^2}} e^{-u^2/2l_{22}^2}}^{f_u(u)}}_{-\frac{df_u(u)}{du} = -f_u'(u)} du$$

$$= -R_{XX}(\tau) \int_{-\infty}^{+\infty} g(u) f_u'(u) du,$$

since $A = LL^*$ gives $l_{12} l_{22} = R_{XX}(\tau)$. Hence

$$R_{XY}(\tau) = R_{XX}(\tau) \{ -\underbrace{g(u) f_u(u) |_{-\infty}^{+\infty}}_{0} + \int_{-\infty}^{+\infty} g'(u) f_u(u) du \}$$

$$= R_{XX}(\tau) E\{g'(X)\} = \eta R_{XX}(\tau),$$

the desired result, where $\eta = E[g'(X)].$ Thus if the input to a memoryless device is stationary Gaussian, the cross correlation function between the input and the output is proportional to the input autocorrelation function.

Linear Systems: $L[\cdot]$ represents a linear system if

$$L\{a_1 X(t_1) + a_2 X(t_2)\} = a_1 L\{X(t_1)\} + a_2 L\{X(t_2)\}. \qquad (12\text{-}28)$$

Let

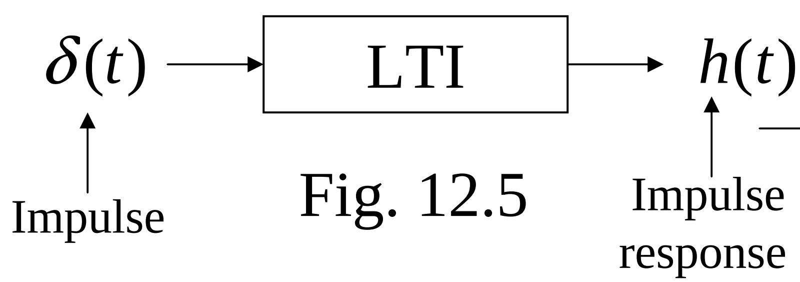
$$Y(t) = L\{X(t)\} \qquad (12\text{-}29)$$

represent the output of a linear system.

Time-Invariant System: $L[\cdot]$ represents a time-invariant system if

$$Y(t) = L\{X(t)\} \Rightarrow L\{X(t-t_0)\} = Y(t-t_0) \qquad (12\text{-}30)$$

i.e., shift in the input results in the same shift in the output also.
If $L[\cdot]$ satisfies both (12-28) and (12-30), then it corresponds to a linear time-invariant (LTI) system.
LTI systems can be uniquely represented in terms of their output to a delta function



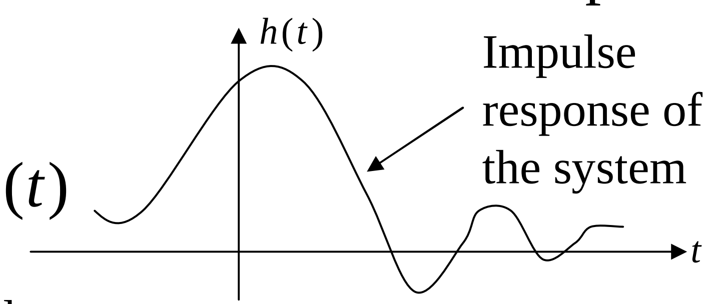


Fig. 12.5

then

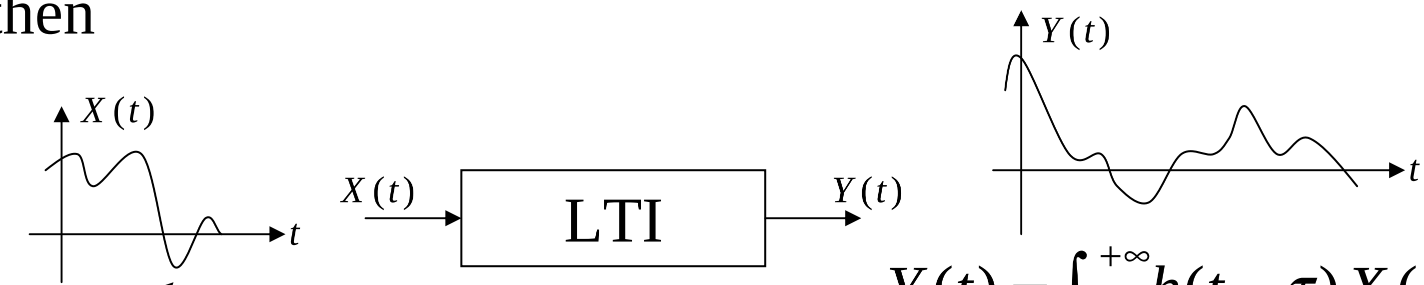


Fig. 12.6

$$Y(t)=\int_{-\infty}^{+\infty}h(t-\tau)X(\tau)d\tau$$

$$=\int_{-\infty}^{+\infty}h(\tau)X(t-\tau)d\tau \quad (12\text{-}31)$$

Eq. (12-31) follows by expressing $X(t)$ as

$$X(t)=\int_{-\infty}^{+\infty}X(\tau)\delta(t-\tau)d\tau \qquad (12\text{-}32)$$

and applying (12-28) and (12-30) to $Y(t)=L\{X(t)\}$. Thus

$$Y(t)=L\{X(t)\}=L\{\int_{-\infty}^{+\infty}X(\tau)\delta(t-\tau)d\tau\}$$

$$=\int_{-\infty}^{+\infty}L\{X(\tau)\delta(t-\tau)d\tau\}$$

By Linearity

$$=\int_{-\infty}^{+\infty}X(\tau)L\{\delta(t-\tau)\}d\tau$$

By Time-invariance

$$=\int_{-\infty}^{+\infty}X(\tau)h(t-\tau)d\tau=\int_{-\infty}^{+\infty}h(\tau)X(t-\tau)d\tau. \quad (12\text{-}33)$$

Output Statistics: Using (12-33), the mean of the output process is given by

$$\mu_Y(t) = E\{Y(t)\} = \int_{-\infty}^{+\infty} E\{X(\tau)h(t-\tau)d\tau\}$$

$$= \int_{-\infty}^{+\infty} \mu_X(\tau)h(t-\tau)d\tau = \mu_X(t) * h(t). \qquad (12\text{-}34)$$

Similarly the cross-correlation function between the input and output processes is given by

$$R_{XY}(t_1,t_2) = E\{X(t_1)Y^*(t_2)\}$$

$$= E\{X(t_1)\int_{-\infty}^{+\infty} X^*(t_2-\alpha)h^*(\alpha)d\alpha\}$$

$$= \int_{-\infty}^{+\infty} E\{X(t_1)X^*(t_2-\alpha)\}h^*(\alpha)d\alpha$$

$$= \int_{-\infty}^{+\infty} R_{XX}(t_1,t_2-\alpha)h^*(\alpha)d\alpha$$

$$= R_{XX}(t_1,t_2) * h^*(t_2). \qquad (12\text{-}35)$$

Finally the output autocorrelation function is given by

$$
\begin{aligned}
R_{YY}(t_1,t_2) &= E\{Y(t_1)Y^*(t_2)\} \\
&= E\{\int_{-\infty}^{+\infty} X(t_1-\beta)h(\beta)d\beta\, Y^*(t_2)\} \\
&= \int_{-\infty}^{+\infty} E\{X(t_1-\beta)Y^*(t_2)\}h(\beta)d\beta \\
&= \int_{-\infty}^{+\infty} R_{XY}(t_1-\beta,t_2)h(\beta)d\beta \\
&= R_{XY}(t_1,t_2) * h(t_1),
\end{aligned}
\qquad (12\text{-}36)
$$

or

$$
R_{YY}(t_1,t_2) = R_{XX}(t_1,t_2) * h^*(t_2) * h(t_1). \qquad (12\text{-}37)
$$

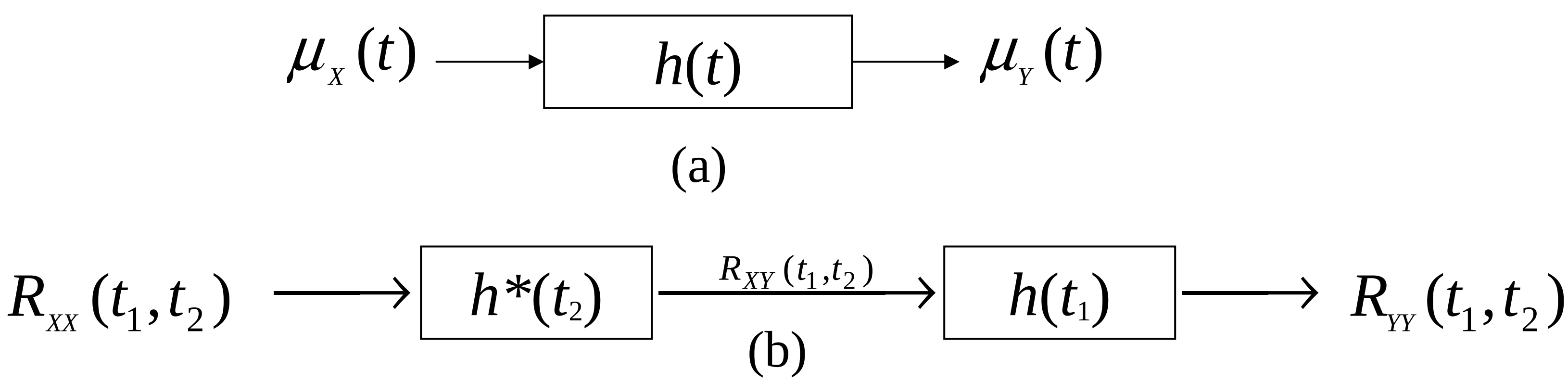


Fig. 12.7

In particular if $X(t)$ is wide-sense stationary, then we have $\mu_X(t) = \mu_X$ so that from (12-34)

$$\mu_Y(t) = \mu_X \int_{-\infty}^{+\infty} h(\tau)d\tau = \mu_X c, \quad a\ constant. \qquad (12\text{-}38)$$

Also $R_{XX}(t_1, t_2) = R_{XX}(t_1 - t_2)$ so that (12-35) reduces to

$$R_{XY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XX}(t_1 - t_2 + \alpha)h^*(\alpha)d\alpha$$
$$= R_{XX}(\tau) * h^*(-\tau) \stackrel{\Delta}{=} R_{XY}(\tau), \quad \tau = t_1 - t_2. \qquad (12\text{-}39)$$

Thus $X(t)$ and $Y(t)$ are jointly w.s.s. Further, from (12-36), the output autocorrelation simplifies to

$$R_{YY}(t_1, t_2) = \int_{-\infty}^{+\infty} R_{XY}(t_1 - \beta - t_2)h(\beta)d\beta, \quad \tau = t_1 - t_2$$
$$= R_{XY}(\tau) * h(\tau) = R_{YY}(\tau). \qquad (12\text{-}40)$$

From (12-37), we obtain

$$R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau). \qquad (12\text{-}41)$$

From (12-38)-(12-40), the output process is also wide-sense stationary. This gives rise to the following representation

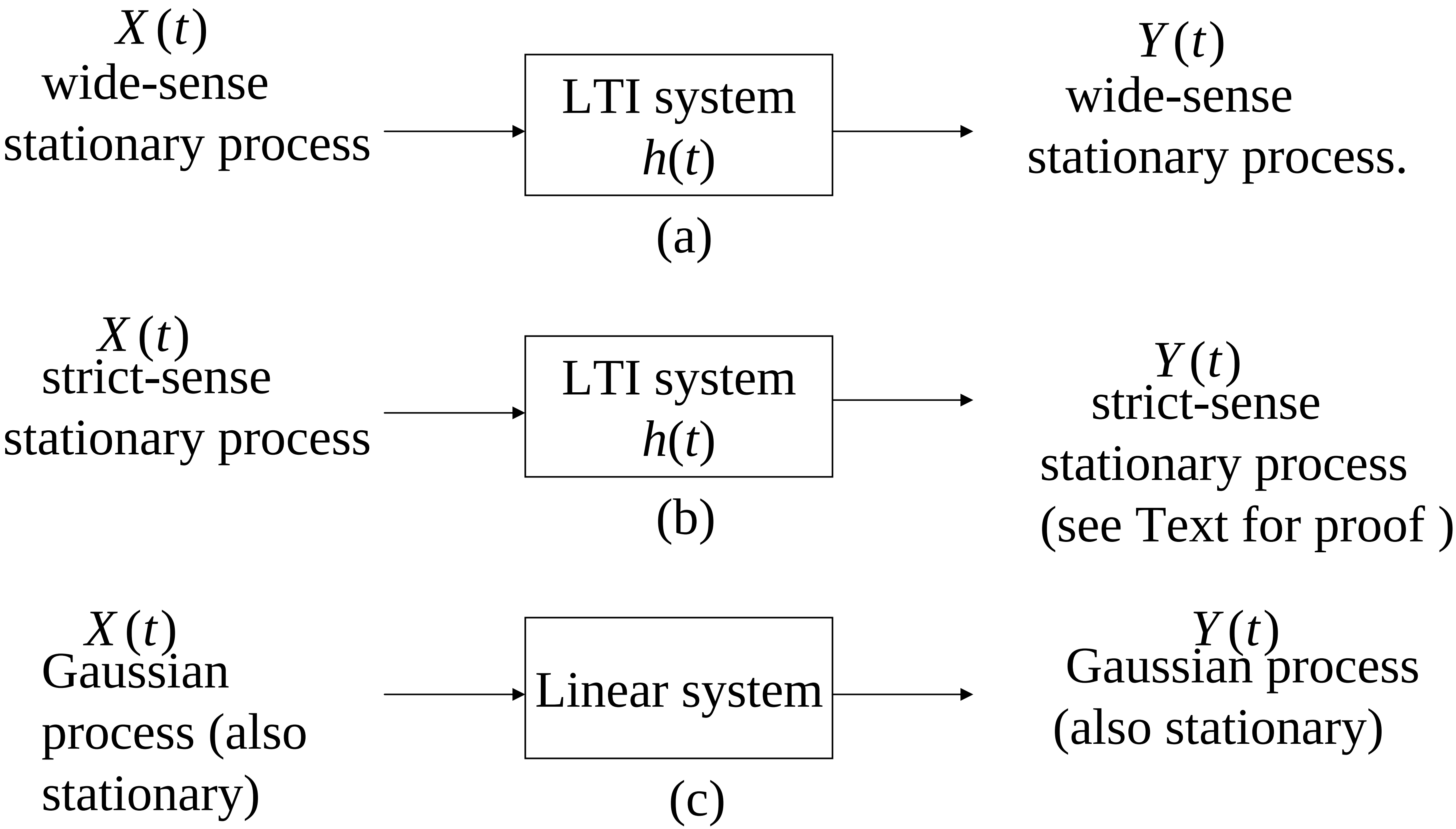


Fig. 12.8

White Noise Process:

$W(t)$ is said to be a white noise process if

$$R_{ww}(t_1,t_2) = q(t_1)\delta(t_1 - t_2), \tag{12-42}$$

i.e., $E[W(t_1)\, W^*(t_2)] = 0$ unless $t_1 = t_2$.

$W(t)$ is said to be wide-sense stationary (w.s.s) white noise if $E[W(t)]$ = constant, and

$$R_{ww}(t_1,t_2) = q\delta(t_1 - t_2) = q\delta(\tau). \tag{12-43}$$

If $W(t)$ is also a Gaussian process (white Gaussian process), then all of its samples are independent random variables (why?).

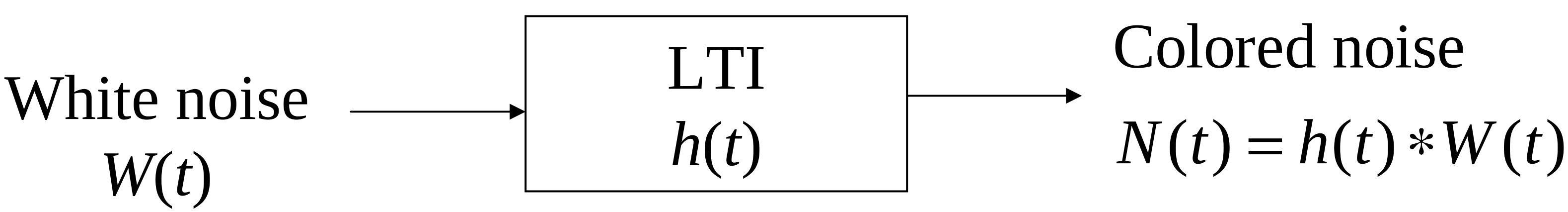


Fig. 12.9

For w.s.s. white noise input $W(t)$, we have

$$E[N(t)] = \mu_W \int_{-\infty}^{+\infty} h(\tau)d\tau, \quad a\ constant \tag{12-44}$$

and

$$\begin{aligned} R_{nn}(\tau) &= q\delta(\tau) * h^*(-\tau) * h(\tau) \\ &= qh^*(-\tau) * h(\tau) = q\rho(\tau) \end{aligned} \tag{12-45}$$

where

$$\rho(\tau) = h(\tau) * h^*(-\tau) = \int_{-\infty}^{+\infty} h(\alpha)h^*(\alpha+\tau)d\alpha. \tag{12-46}$$

Thus the output of a white noise process through an LTI system represents a (colored) noise process.

Note: White noise need not be Gaussian.

“White” and “Gaussian” are two different concepts!