13. Power Spectrum

For a deterministic signal x(t), the spectrum is well defined: If $X(\omega)$ represents its Fourier transform, i.e., if

$$X(\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt, \qquad (13\text{-}1)$$

then $|X(\omega)|^2$ represents its energy spectrum. This follows from Parseval's theorem since the signal energy is given by

$$\int_{-\infty}^{+\infty} x^2(t) dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X(\omega)|^2 \, d\omega = E. \qquad (13\text{-}2)$$

Thus $|X(\omega)|^2 \Delta\omega$ represents the signal energy in the band $(\omega, \omega + \Delta\omega)$ (see Fig 13.1).

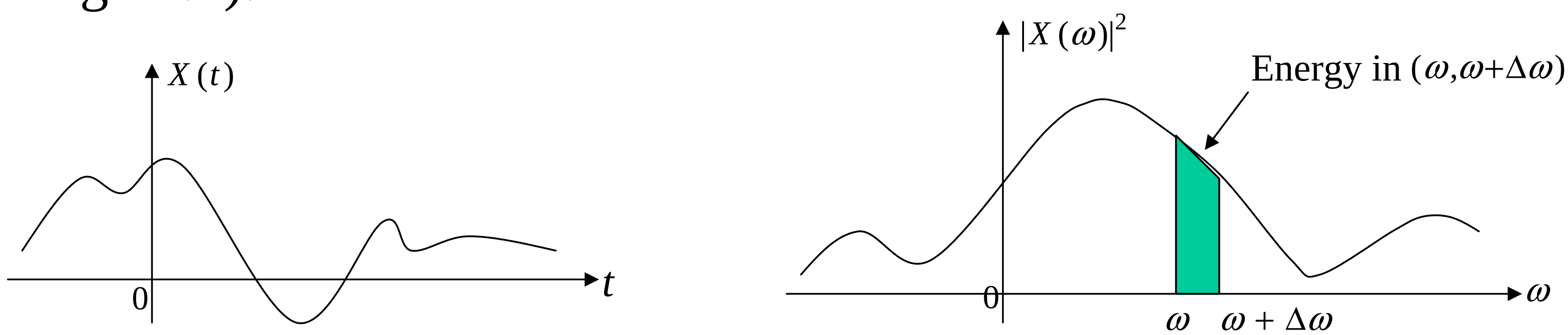


Fig 13.1

However for stochastic processes, a direct application of (13-1) generates a sequence of random variables for every ω. Moreover, for a stochastic process, $E\{|X(t)|^2\}$ represents the ensemble average power (instantaneous energy) at the instant t.

To obtain the spectral distribution of power versus frequency for stochastic processes, it is best to avoid infinite intervals to begin with, and start with a finite interval $(-T, \; T)$ in (13-1). Formally, partial Fourier transform of a process $X(t)$ based on $(-T, \; T)$ is given by

$$X_T(\omega) = \int_{-T}^{T} X(t) e^{-j\omega t} dt \tag{13-3}$$

so that

$$\frac{|X_T(\omega)|^2}{2T} = \frac{1}{2T} \left| \int_{-T}^{T} X(t) e^{-j\omega t} dt \right|^2 \tag{13-4}$$

represents the power distribution associated with that realization based on $(-T, \; T)$. Notice that (13-4) represents a random variable for every ω, and its ensemble average gives, the average power distribution based on $(-T, \; T)$. Thus

$$P_T(\omega) = E\left\{\frac{|X_T(\omega)|^2}{2T}\right\} = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} E\{X(t_1)X^*(t_2)\}e^{-j\omega(t_1-t_2)}dt_1dt_2$$

$$= \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1,t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2 \qquad (13\text{-}5)$$

represents the power distribution of $X(t)$ based on $(-T,\ T)$. For wide sense stationary (w.s.s) processes, it is possible to further simplify (13-5). Thus if $X(t)$ is assumed to be w.s.s, then $R_{XX}(t_1,t_2) = R_{XX}(t_1 - t_2)$ and (13-5) simplifies to

$$P_T(\omega) = \frac{1}{2T}\int_{-T}^{T}\int_{-T}^{T} R_{XX}(t_1 - t_2)e^{-j\omega(t_1-t_2)}dt_1dt_2.$$

Let $\tau = t_1 - t_2$ and proceeding as in (14-24), we get

$$P_T(\omega) = \frac{1}{2T}\int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(2T - |\tau|)d\tau$$

$$= \int_{-2T}^{2T} R_{XX}(\tau)e^{-j\omega\tau}(1 - \tfrac{|\tau|}{2T})d\tau \ \geq\ 0 \qquad (13\text{-}6)$$

to be the power distribution of the w.s.s. process $X(t)$ based on $(-T, T)$. Finally letting $T \to \infty$ in (13-6), we obtain

$$S_{XX}(\omega) = \lim_{T\to\infty} P_T(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau \geq 0 \qquad (13\text{-}7)$$

to be the *power spectral density* of the w.s.s process *X*(*t*). Notice that

$$R_{XX}(\omega) \xleftrightarrow{\text{F}\cdot\text{T}} S_{XX}(\omega) \geq 0. \qquad (13\text{-}8)$$

i.e., the autocorrelation function and the power spectrum of a w.s.s Process form a Fourier transform pair, a relation known as the **Wiener-Khinchin Theorem**. From (13-8), the inverse formula gives

$$R_{XX}(\tau) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega\tau} d\omega \qquad (13\text{-}9)$$

and in particular for $\tau = 0,$ we get

$$\frac{1}{2\pi} \int_{-\infty}^{+\infty} S_{XX}(\omega) d\omega = R_{XX}(0) = E\{|X(t)|^2\} = P, \quad \textit{the total power.}$$

(13-10)

From (13-10), the area under $S_{XX}(\omega)$ represents the total power of the process *X*(*t*), and hence $S_{XX}(\omega)$ truly represents the power spectrum. (Fig 13.2).

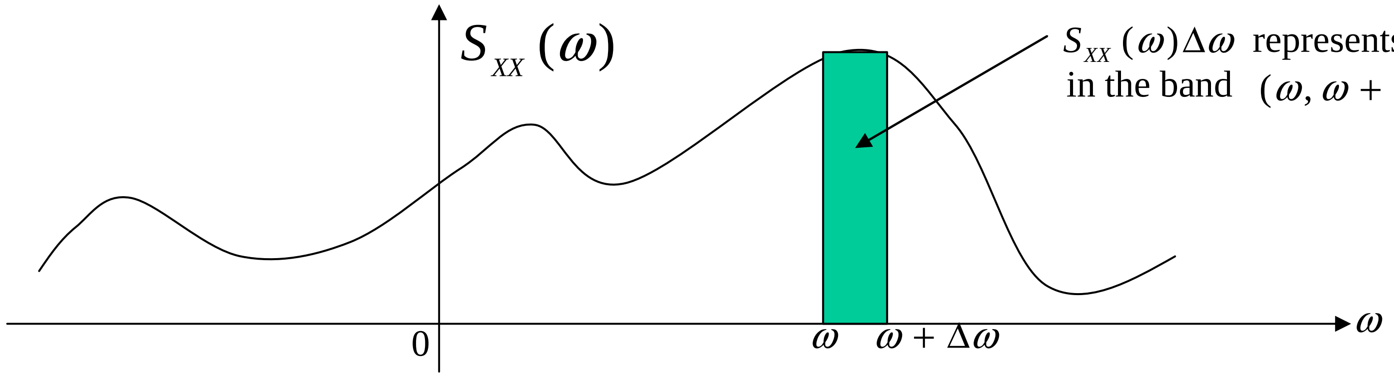


Fig 13.2

The nonnegative-definiteness property of the autocorrelation function in (14-8) translates into the "nonnegative" property for its Fourier transform (power spectrum), since from (14-8) and (13-9)

$$\sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* R_{XX}(t_i - t_j) = \sum_{i=1}^{n}\sum_{j=1}^{n} a_i a_j^* \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) e^{j\omega(t_i - t_j)} d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} S_{XX}(\omega) \left| \sum_{i=1}^{n} a_i e^{j\omega t_i} \right|^2 d\omega \geq 0. \quad (13\text{-}11)$$

From (13-11), it follows that

$$R_{XX}(\tau) \ \textit{nonnegative-definite} \quad \Leftrightarrow \quad S_{XX}(\omega) \geq 0. \qquad (13\text{-}12)$$

If $X(t)$ is a real w.s.s process, then $R_{XX}(\tau) = R_{XX}(-\tau)$ so that

$$S_{XX}(\omega) = \int_{-\infty}^{+\infty} R_{XX}(\tau) e^{-j\omega\tau} d\tau$$

$$= \int_{-\infty}^{+\infty} R_{XX}(\tau) \cos \omega\tau d\tau$$

$$= 2\int_{0}^{\infty} R_{XX}(\tau) \cos \omega\tau d\tau = S_{XX}(-\omega) \geq 0 \qquad (13\text{-}13)$$

so that the power spectrum is an even function, (in addition to being real and nonnegative).

Power Spectra and Linear Systems

If a w.s.s process $X(t)$ with autocorrelation function $R_{XX}(\tau) \;\leftrightarrow\; S_{XX}(\tau) \geq 0$ is applied to a linear system with impulse response $h(t)$, then the cross correlation function $R_{XY}(\tau)$ and the output autocorrelation function $R_{YY}(\tau)$ are given by (14-40)-(14-41). From there

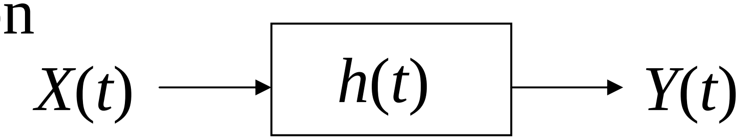


Fig 13.3

$$R_{XY}(\tau) = R_{XX}(\tau) * h^*(-\tau), \quad R_{YY}(\tau) = R_{XX}(\tau) * h^*(-\tau) * h(\tau). \qquad (13\text{-}14)$$

But if

$$f(t) \;\leftrightarrow\; F(\omega), \quad g(t) \;\leftrightarrow\; G(\omega) \qquad (13\text{-}15)$$

Then

$$f(t) * g(t) \;\leftrightarrow\; F(\omega)G(\omega) \qquad (13\text{-}16)$$

since

$$\mathcal{F}\{f(t) * g(t)\} = \int_{-\infty}^{+\infty} f(t) * g(t) e^{-j\omega t} dt$$

$$\mathcal{F}\{f(t)*g(t)\}=\int_{-\infty}^{+\infty}\left\{\int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau\right\}e^{-j\omega t}dt$$

$$=\int_{-\infty}^{+\infty} f(\tau)e^{-j\omega\tau}d\tau \int_{-\infty}^{+\infty} g(t-\tau)e^{-j\omega(t-\tau)}d(t-\tau)$$

$$=F(\omega)G(\omega). \qquad (13\text{-}17)$$

Using (13-15)-(13-17) in (13-14) we get

$$S_{XY}(\omega)=\mathcal{F}\{R_{XX}(\omega)*h^*(-\tau)\}=S_{XX}(\omega)H^*(\omega) \qquad (13\text{-}13)$$

since

$$\int_{-\infty}^{+\infty} h^*(-\tau)e^{-j\omega\tau}d\tau=\left(\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt\right)^* = H^*(\omega),$$

where

$$H(\omega)=\int_{-\infty}^{+\infty} h(t)e^{-j\omega t}dt \qquad (13\text{-}19)$$

represents the transfer function of the system, and

$$S_{YY}(\omega)=\mathcal{F}\{R_{YY}(\tau)\}=S_{XY}(\omega)H(\omega)$$

$$=S_{XX}(\omega)\,|\,H(\omega)\,|^2\,. \qquad (13\text{-}20)$$

From (13-13), the cross spectrum need not be real or nonnegative; However the output power spectrum is real and nonnegative and is related to the input spectrum and the system transfer function as in (13-20). Eq. (13-20) can be used for system identification as well.

W.S.S White Noise Process: If $W(t)$ is a w.s.s white noise process, then from (14-43)

$$R_{WW}(\tau) = q\delta(\tau) \quad \Rightarrow \quad S_{WW}(\omega) = q. \tag{13-21}$$

Thus the spectrum of a white noise process is flat, thus justifying its name. Notice that a white noise process is unrealizable since its total power is indeterminate.

From (13-20), if the input to an unknown system in Fig 13.3 is a white noise process, then the output spectrum is given by

$$S_{YY}(\omega) = q\,|\,H(\omega)\,|^2 \tag{13-22}$$

Notice that the output spectrum captures the system transfer function characteristics entirely, and for rational systems Eq (13-22) may be used to determine the pole/zero locations of the underlying system.

Example 13.1: A w.s.s white noise process $W(t)$ is passed through a low pass filter (LPF) with bandwidth $B/2$. Find the autocorrelation function of the output process.

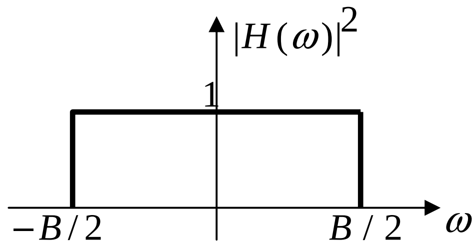
Solution: Let $X(t)$ represent the output of the LPF. Then from (13-22)

$$S_{XX}(\omega) = q\,|H(\omega)|^2 = \begin{cases} q, & |\omega| \le B/2 \\ 0, & |\omega| > B/2 \end{cases} . \qquad (13\text{-}23)$$

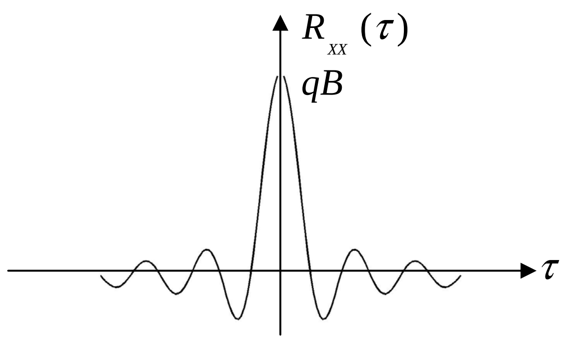
Inverse transform of $S_{XX}(\omega)$ gives the output autocorrelation function to be

$$R_{XX}(\tau) = \int_{-B/2}^{B/2} S_{XX}(\omega) e^{j\omega\tau} d\omega = q\int_{-B/2}^{B/2} e^{j\omega\tau} d\omega$$

$$= qB\,\frac{\sin(B\tau/2)}{(B\tau/2)} = qB\,\text{sinc}(B\tau/2) \qquad (13\text{-}24)$$



(a) LPF



(b)

Fig. 13.4

Eq (13-23) represents colored noise spectrum and (13-24) its autocorrelation function (see Fig 13.4).

Example 13.2: Let

$$Y(t) = \frac{1}{2T}\int_{t-T}^{t+T} X(\tau)d\tau \qquad (13\text{-}25)$$

represent a "smoothing" operation using a moving window on the input process *X*(*t*). Find the spectrum of the output *Y*(*t*) in term of that of *X*(*t*).

Solution: If we define an LTI system with impulse response *h*(*t*) as in Fig 13.5, then in term of *h*(*t*), Eq (13-25) reduces to

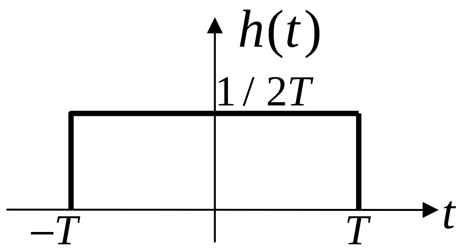


Fig 13.5

$$Y(t) = \int_{-\infty}^{+\infty} h(t-\tau)X(\tau)d\tau = h(t) * X(t) \qquad (13\text{-}26)$$

so that

$$S_{YY}(\omega) = S_{XX}(\omega)\,|\,H(\omega)\,|^2 \,. \qquad (13\text{-}27)$$

Here

$$H(\omega) = \int_{-T}^{+T} \tfrac{1}{2T} e^{-j\omega t} dt = \text{sinc}(\omega T) \qquad (13\text{-}28)$$

so that

$$S_{YY}(\omega) = S_{XX}(\omega)\,\text{sinc}^2(\omega T). \qquad (13\text{-}29)$$

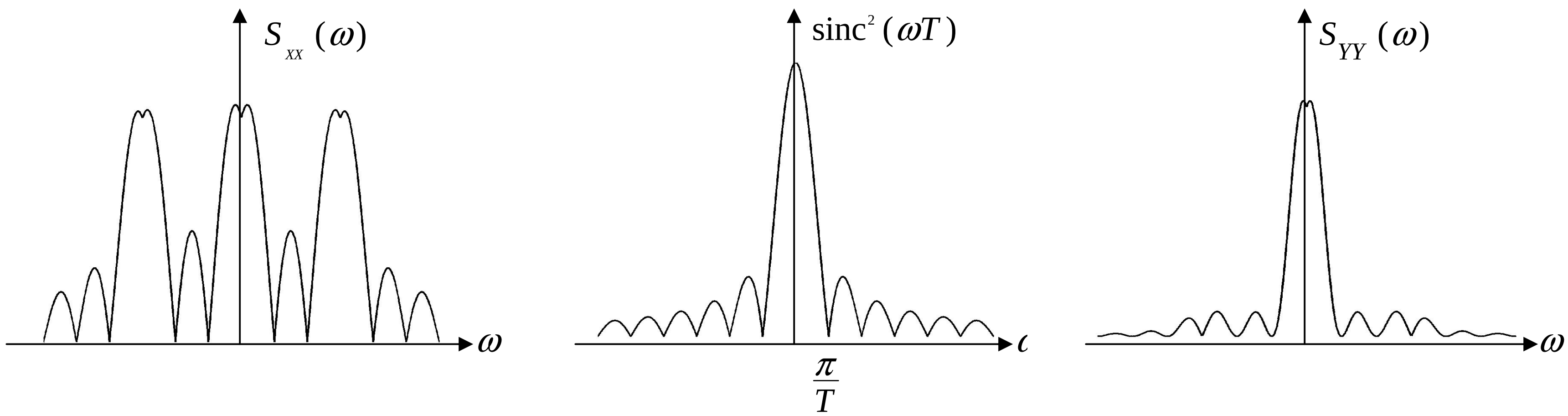


Fig 13.6

Notice that the effect of the smoothing operation in (13-25) is to suppress the high frequency components in the input (beyond π / T), and the equivalent linear system acts as a low-pass filter (continuous-time moving average) with bandwidth $2\pi / T$ in this case.

Discrete – Time Processes

For discrete-time w.s.s stochastic processes $X(nT)$ with autocorrelation sequence $\{r_k\}_{-\infty}^{+\infty}$, (proceeding as above) or formally defining a continuous time process $X(t) = \sum_n X(nT)\delta(t - nT)$, we get the corresponding autocorrelation function to be

$$R_{XX}(\tau) = \sum_{k=-\infty}^{+\infty} r_k \delta(\tau - kT).$$

Its Fourier transform is given by

$$S_{XX}(\omega) = \sum_{k=-\infty}^{+\infty} r_k e^{-j\omega T} \geq 0, \qquad (13\text{-}30)$$

and it defines the power spectrum of the discrete-time process $X(nT)$. From (13-30),

$$S_{XX}(\omega) = S_{XX}(\omega + 2\pi / T) \qquad (13\text{-}31)$$

so that $S_{XX}(\omega)$ is a periodic function with period

$$2B = \frac{2\pi}{T}. \qquad (13\text{-}32)$$

This gives the inverse relation

$$r_k = \frac{1}{2B}\int_{-B}^{B} S_{XX}(\omega) e^{jk\omega T} d\omega \qquad (13\text{-}33)$$

and

$$r_0 = E\{| X(nT)|^2\} = \frac{1}{2B}\int_{-B}^{B} S_{XX}(\omega) d\omega \qquad (13\text{-}34)$$

represents the total power of the discrete-time process $X(nT)$. The input-output relations for discrete-time system $h(nT)$ in (14-65)-(14-67) translate into

$$S_{XY}(\omega) = S_{XX}(\omega) H^*(e^{j\omega}) \qquad (13\text{-}35)$$

and

$$S_{YY}(\omega) = S_{XX}(\omega) | H(e^{j\omega})|^2 \qquad (13\text{-}36)$$

where

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} h(nT)\, e^{-j\omega nT} \qquad (13\text{-}37)$$

represents the discrete-time system transfer function.