14. Poisson Processes

In Lecture 4, we introduced Poisson arrivals as the limiting behavior of Binomial random variables. (Refer to Poisson approximation of Binomial random variables.)
From the discussion there (see (4-6)-(4-8) Lecture 4)

$$P\left\{ \begin{array}{l} \text{"}k \text{ arrivals occur in an} \\ \text{ interval of duration } \Delta\text{"} \end{array} \right\} = e^{-\lambda} \frac{\lambda^k}{k!}, \qquad k = 0, 1, 2, \cdots \qquad (14\text{-}1)$$

where

$$\lambda = np = \mu T \cdot \frac{\Delta}{T} = \mu \Delta \qquad (14\text{-}2)$$

Fig. 14.1

It follows that (refer to Fig. 14.1)

$$P\left\{\begin{array}{l}\text{"}k \text{ arrivals occur in an} \\ \text{interval of duration } 2\Delta\text{"}\end{array}\right\} = e^{-2\lambda}\frac{(2\lambda)^k}{k!}, \qquad k = 0,\ 1,\ 2,\cdots, \qquad (14\text{-}3)$$

since in that case

$$np_1 = \mu T \cdot \frac{2\Delta}{T} = 2\mu\Delta = 2\lambda. \qquad (14\text{-}4)$$

From (14-1)-(14-4), Poisson arrivals over an interval form a Poisson random variable whose parameter depends on the duration of that interval. Moreover because of the Bernoulli nature of the underlying basic random arrivals, events over nonoverlapping intervals are independent. We shall use these two key observations to define a Poisson process formally. (Refer to Example 9-5, Text)

Definition: $X(t) = n(0, t)$ represents a Poisson process if

(i) the number of arrivals $n(t_1, t_2)$ in an interval (t_1, t_2) of length $t = t_2 - t_1$ is a Poisson random variable with parameter λt. Thus

$$P\{n(t_1, t_2) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \cdots, \quad t = t_2 - t_1 \tag{14-5}$$

and

(ii) If the intervals (t_1, t_2) and (t_3, t_4) are nonoverlapping, then the random variables $n(t_1, t_2)$ and $n(t_3, t_4)$ are independent.

Since $n(0, t) \sim P(\lambda t)$, we have

$$E[X(t)] = E[n(0,t)] = \lambda t \tag{14-6}$$

and

$$E[X^2(t)] = E[n^2(0,t)] = \lambda t + \lambda^2 t^2. \tag{14-7}$$

To determine the autocorrelation function $R_{XX}(t_1, t_2)$, let $t_2 > t_1$, then from (ii) above $n(0, t_1)$ and $n(t_1, t_2)$ are independent Poisson random variables with parameters λt_1 and $\lambda(t_2 - t_1)$ respectively. Thus

$$E[n(0,t_1)n(t_1,t_2)] = E[n(0,t_1)]E[n(t_1,t_2)] = \lambda^2 t_1(t_2 - t_1). \tag{14-8}$$

But

$$n(t_1, t_2) = n(0, t_2) - n(0, t_1) = X(t_2) - X(t_1)$$

and hence the left side if (14-8) can be rewritten as

$$E[X(t_1)\{X(t_2) - X(t_1)\}] = R_{XX}(t_1, t_2) - E[X^2(t_1)]. \qquad (14\text{-}9)$$

Using (14-7) in (14-9) together with (14-8), we obtain

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 (t_2 - t_1) + E[X^2(t_1)]$$

$$= \lambda t_1 + \lambda^2 t_1 \, t_2, \qquad t_2 \geq t_1 \, . \qquad (14\text{-}10)$$

Similarly

$$R_{XX}(t_1, t_2) = \lambda t_2 + \lambda^2 t_1 \, t_2, \qquad t_2 < t_1 \, . \qquad (14\text{-}11)$$

Thus

$$R_{XX}(t_1, t_2) = \lambda^2 t_1 \, t_2 + \lambda \min(t_1, t_2). \qquad (14\text{-}12)$$

From (14-12), notice that the Poisson process $X(t)$ *does not* represent a wide sense stationary process.

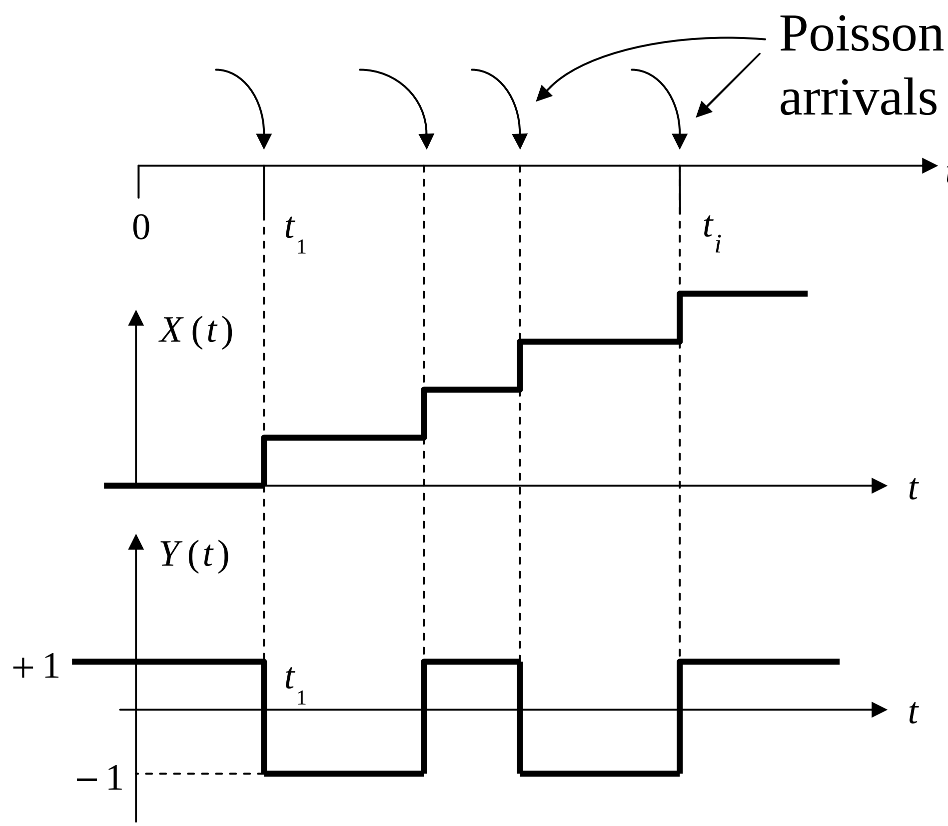

Fig. 14.2

Define a binary level process

$$Y(t) = (-1)^{X(t)} \tag{14-13}$$

that represents a telegraph signal (Fig. 14.2). Notice that the transition instants $\{t_i\}$ are random. (see Example 9-6, Text for the mean and autocorrelation function of a telegraph signal). Although $X(t)$ does not represent a wide sense stationary process, its derivative $X'(t)$ *does* represent a wide sense stationary process.

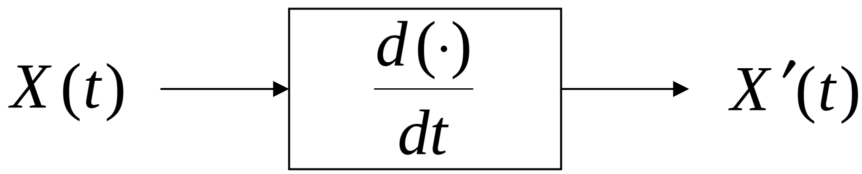

Fig. 14.3 (Derivative as a LTI system)

To see this, we can make use of Fig. 14.7 and (14-34)-(14-37). From there

$$\mu_{X'}(t) = \frac{d\mu_X(t)}{dt} = \frac{d\lambda t}{dt} = \lambda, \quad a\ constant \tag{14-14}$$

and

$$R_{XX'}(t_1,t_2) = \frac{\partial\, R_{XX}(t_1,\ t_2)}{\partial\ t_2} = \begin{cases} \lambda^2 t_1 & t_1 \le t_2 \\ \lambda^2 t_1 + \lambda & t_1 > t_2 \end{cases}$$

$$= \lambda^2 t_1 + \lambda\ U(t_1 - t_2) \tag{14-14}$$

and

$$R_{X'X'}(t_1,\ t_2) = \frac{\partial\, R_{XX'}(t_1,t_2)}{\partial\, t_1} = \lambda^2 + \lambda\, \delta(t_1 -\ t_2). \tag{14-16}$$

From (14-14) and (14-16) it follows that $X'(t)$ is a wide sense stationary process. Thus nonstationary inputs to linear systems *can* lead to wide sense stationary outputs, an interesting observation.

- **Sum of Poisson Processes:**

  If $X_1(t)$ and $X_2(t)$ represent two independent Poisson processes, then their sum $X_1(t) + X_2(t)$ is also a Poisson process with parameter $(\lambda_1 + \lambda_2)t$. (Follows from (6-86), Text and the definition of the Poisson process in (i) and (ii)).

- **Random selection of Poisson Points:**

  Let $t_1, t_2, \cdots, t_i, \cdots$ represent random arrival points associated with a Poisson process $X(t)$ with parameter λt, and associated with each arrival point, define an independent Bernoulli random variable N_i, where

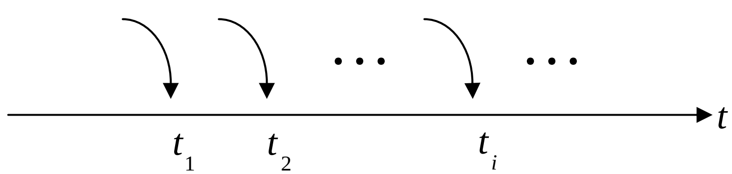

Fig. 14.4

$$P(N_i = 1) = p, \qquad P(N_i = 0) = q = 1 - p. \qquad (14\text{-}17)$$

Define the processes

$$Y(t)=\sum_{i=1}^{X(t)} N_i \quad ; \quad Z(t)=\sum_{i=1}^{X(t)} (1-N_i)=X(t)-Y(t) \qquad (14\text{-}18)$$

we claim that both $Y(t)$ and $Z(t)$ are independent Poisson processes with parameters λpt and λqt respectively.

Proof:

$$Y(t)=\sum_{n=k}^{\infty} P\{Y(t)=k \mid X(t)=n\}P\{X(t)=n)\}. \qquad (14\text{-}19)$$

But given $X(t) = n$, we have $Y(t)=\sum_{i=1}^{n} N_i \sim B(n,p)$ so that

$$P\{Y(t)=k \mid X(t)=n\}=\binom{n}{k} p^k q^{n-k}, \qquad 0 \le k \le n, \qquad (14\text{-}20)$$

and

$$P\{X(t)=n\}=e^{-\lambda t}\frac{(\lambda t)^n}{n!}. \qquad (14\text{-}21)$$

Substituting (14-20)-(14-21) into (14-19) we get

$$P\{Y(t)=k\}=e^{-\lambda t}\sum_{n=k}^{\infty}\frac{n!}{(n-k)!k!}p^k q^{n-k}\frac{(\lambda t)^n}{n!}=\frac{p^k e^{-\lambda t}}{k!}(\lambda t)^k\underbrace{\sum_{n=k}^{\infty}\frac{(q\lambda t)^{n-k}}{(n-k)!}}_{e^{q\lambda t}}$$

$$=(\lambda pt)^k\frac{e^{-(1-q)\lambda t}}{k!}=e^{-\lambda pt}\frac{(\lambda pt)^k}{k!},\qquad k=0,1,2,\cdots$$

$$\sim\quad P(\lambda pt). \qquad (14\text{-}22)$$

More generally,

$$P\{Y(t)=k,Z(t)=m\}=P\{Y(t)=k,X(t)-Y(t)=m\}$$

$$=P\{Y(t)=k,X(t)=k+m\}$$

$$=P\{Y(t)=k\mid X(t)=k+m\}P\{X(t)=k+m\}$$

$$=\binom{k+m}{k}p^k q^m\cdot e^{-\lambda t}\frac{(\lambda t)^{k+m}}{(k+m)!}=\underbrace{e^{-\lambda pt}\frac{(\lambda pt)^n}{k!}}_{P(Y(t)=k)}\underbrace{e^{-\lambda qt}\frac{(\lambda qt)^n}{m!}}_{P(Z(t)=m)}$$

$$=P\{Y(t)=k\}P\{Z(t)=m\}, \qquad (14\text{-}23)$$

which completes the proof.

Notice that $Y(t)$ and $Z(t)$ are generated as a result of random Bernoulli selections from the original Poisson process $X(t)$ (Fig. 14.5), where each arrival gets tossed over to either $Y(t)$ with probability p or to $Z(t)$ with probability q. Each such sub-arrival stream is also a Poisson process. Thus random selection of Poisson points preserve the Poisson nature of the resulting processes. However, as we shall see deterministic selection from a Poisson process destroys the Poisson property for the resulting processes.

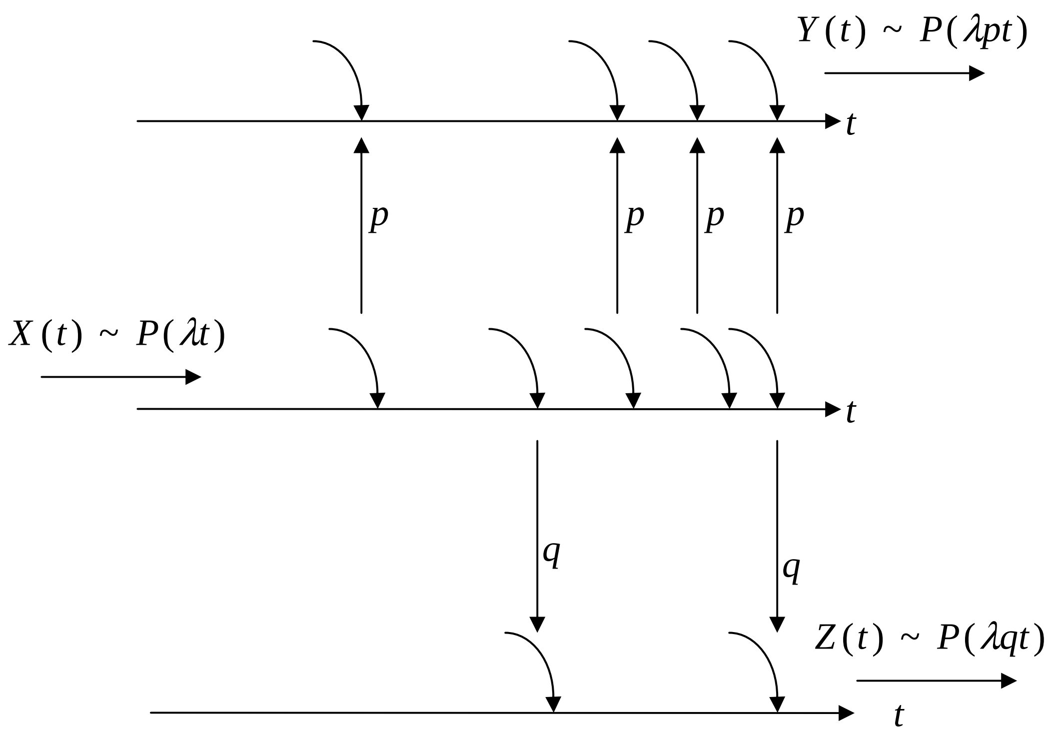

Fig. 14.5

Inter-arrival Distribution for Poisson Processes

Let τ_1 denote the time interval (delay) to the first arrival from *any* fixed point t_0. To determine the probability distribution of the random variable τ_1, we argue as follows: Observe that

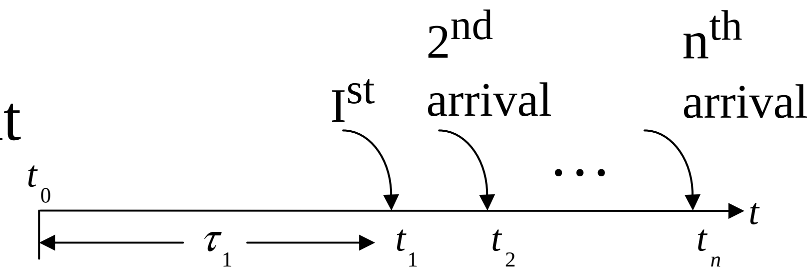

Fig. 14.6

the event $"\tau_1 > t"$ is the same as "$n(t_0, t_0+t) = 0$", or the complement event $"\tau_1 \le t"$ is the same as the event "$n(t_0, t_0+t) > 0$" .
Hence the distribution function of τ_1 is given by

$$F_{\tau_1}(t) \triangleq P\{\tau_1 \le t\} = P\{X(t) > 0\} = P\{n(t_0, t_0 + t) > 0\}$$

$$= 1 - P\{n(t_0, t_0 + t) = 0\} = 1 - e^{-\lambda t} \qquad (14\text{-}24)$$

(use (14-5)), and hence its derivative gives the probability density function for τ_1 to be

$$f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, \qquad t \ge 0 \qquad (14\text{-}25)$$

i.e., τ_1 is an exponential random variable with parameter λ so that $E(\tau_1) = 1/\lambda.$

Similarly, let t_n represent the n^{th} random arrival point for a Poisson process. Then

$$F_{t_n}(t) \triangleq P\{t_n \le t\} = P\{X(t) \ge n\}$$

$$= 1 - P\{X(t) < n\} = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t} \qquad (14\text{-}26)$$

and hence

$$f_{t_n}(x) = \frac{dF_{t_n}(x)}{dx} = -\sum_{k=1}^{n-1} \frac{\lambda(\lambda x)^{k-1}}{(k-1)!} e^{-\lambda x} + \sum_{k=0}^{n-1} \frac{\lambda(\lambda x)^k}{k!} e^{-\lambda x}$$

$$= \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, \qquad x \ge 0 \qquad (14\text{-}27)$$

which represents a gamma density function. i.e., the waiting time to the n^{th} Poisson arrival instant has a gamma distribution. Moreover

$$t_n = \sum_{i=1}^{n} \tau_i$$

where τ_i is the random inter-arrival duration between the $(i-1)^{th}$ and i^{th} events. Notice that τ_i s are independent, identically distributed random variables. Hence using their characteristic functions, it follows that all inter-arrival durations of a Poisson process are independent exponential random variables with common parameter λ. i.e.,

$$f_{\tau_i}(t) = \lambda e^{-\lambda t}, \qquad t \geq 0. \tag{14-28}$$

Alternatively, from (14-24)-(14-25), we have τ_1 is an exponential random variable. By repeating that argument after shifting t_0 to the new point t_1 in Fig. 14.6, we conclude that τ_2 is an exponential random variable. Thus the sequence $\tau_1, \tau_2, \cdots, \tau_n, \cdots$ are independent exponential random variables with common p.d.f as in (14-25).

Thus if we systematically tag every m^{th} outcome of a Poisson process $X(t)$ with parameter λt to generate a new process $e(t)$, then the inter-arrival time between any two events of $e(t)$ is a gamma random variable.

Notice that

$$E[e(t)] = m/\lambda, \text{ and if } \lambda = m\mu, \text{ then } E[e(t)] = 1/\mu.$$

The inter-arrival time of $e(t)$ in that case represents an Erlang-m random variable, and $e(t)$ an Erlang-m process (see (10-90), Text). In summary, if Poisson arrivals are randomly redirected to form new queues, then each such queue generates a new Poisson process (Fig. 14.5). However if the arrivals are systematically redirected (Ist arrival to Ist counter, 2nd arrival to 2nd counter, $\cdots$, m^{th} to m^{th}, $(m+1)^{st}$ arrival to Ist counter, $\cdots$), then the new subqueues form Erlang-m processes.

Interestingly, we can also derive the key Poisson properties (14-5) and (14-25) by starting from a simple axiomatic approach as shown below: