14. Poisson Processes

In Lecture 4, we introduced Poisson arrivals as the limiting behavior of Binomial random variables. (Refer to Poisson approximation of Binomial random variables.)

From the discussion there (see (4-6)-(4-8) Lecture 4)

\[
P\left\{"k \text{ arrivals occur in an interval of duration } \Delta"\right\} = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots \quad (14-1)
\]

where

\[
\lambda = np = \mu T \cdot \frac{\Delta}{T} = \mu \Delta \quad (14-2)
\]

Fig. 14.1
It follows that (refer to Fig. 14.1)

\[
P \left\{ \begin{array}{l}
\text{"} k \text{ arrivals occur in an} \\
\text{interval of duration } 2\Delta"
\end{array} \right\} = e^{-2\lambda} \frac{(2\lambda)^k}{k!}, \quad k = 0, 1, 2, \cdots,
\]

since in that case

\[
np_1 = \mu T \cdot \frac{2\Delta}{T} = 2\mu\Delta = 2\lambda.
\]

From (14-1)-(14-4), Poisson arrivals over an interval form a Poisson random variable whose parameter depends on the duration of that interval. Moreover because of the Bernoulli nature of the underlying basic random arrivals, events over nonoverlapping intervals are independent. We shall use these two key observations to define a Poisson process formally. (Refer to Example 9-5, Text)

**Definition:** \( X(t) = n(0, t) \) represents a Poisson process if

(i) the number of arrivals \( n(t_1, t_2) \) in an interval \( (t_1, t_2) \) of length \( t = t_2 - t_1 \) is a Poisson random variable with parameter \( \lambda t \).
\[ P\{n(t_1, t_2) = k\} = e^{-\lambda t} \frac{(\lambda t)^k}{k!}, \quad k = 0, 1, 2, \ldots, \quad t = t_2 - t_1 \quad (14-5) \]

and

(ii) If the intervals \((t_1, t_2)\) and \((t_3, t_4)\) are nonoverlapping, then the random variables \(n(t_1, t_2)\) and \(n(t_3, t_4)\) are independent.

Since \(n(0, t) \sim P(\lambda t)\), we have

\[ E[X(t)] = E[n(0, t)] = \lambda t \quad (14-6) \]

and

\[ E[X^2(t)] = E[n^2(0, t)] = \lambda t + \lambda^2 t^2. \quad (14-7) \]

To determine the autocorrelation function \(R_{xx}(t_1, t_2)\), let \(t_2 > t_1\), then from (ii) above \(n(0, t_1)\) and \(n(t_1, t_2)\) are independent Poisson random variables with parameters \(\lambda t_1\) and \(\lambda(t_2 - t_1)\) respectively. Thus

\[ E[n(0, t_1)n(t_1, t_2)] = E[n(0, t_1)]E[n(t_1, t_2)] = \lambda^2 t_1(t_2 - t_1). \quad (14-8) \]
But
\[ n(t_1, t_2) = n(0, t_2) - n(0, t_1) = X(t_2) - X(t_1) \]
and hence the left side if (14-8) can be rewritten as
\[ E[X(t_1) \{ X(t_2) - X(t_1) \}] = R_{xx}(t_1, t_2) - E[X^2(t_1)]. \quad (14-9) \]
Using (14-7) in (14-9) together with (14-8), we obtain
\[ R_{xx}(t_1, t_2) = \lambda^2 t_1 (t_2 - t_1) + E[X^2(t_1)] \]
\[ = \lambda t_1 + \lambda^2 t_1 t_2, \quad t_2 \geq t_1. \quad (14-10) \]
Similarly
\[ R_{xx}(t_1, t_2) = \lambda t_2 + \lambda^2 t_1 t_2, \quad t_2 < t_1. \quad (14-11) \]
Thus
\[ R_{xx}(t_1, t_2) = \lambda^2 t_1 t_2 + \lambda \min(t_1, t_2). \quad (14-12) \]
From (14-12), notice that the Poisson process $X(t)$ does not represent a wide sense stationary process.

Define a binary level process

$$Y(t) = (-1)^{X(t)}$$

(14-13)

that represents a telegraph signal (Fig. 14.2). Notice that the transition instants $\{t_i\}$ are random. (see Example 9-6, Text for the mean and autocorrelation function of a telegraph signal). Although $X(t)$ does not represent a wide sense stationary process,
its derivative $X'(t)$ does represent a wide sense stationary process.

$$X(t) \xrightarrow{\frac{d}{dt}} X'(t)$$

Fig. 14.3 (Derivative as a LTI system)

To see this, we can make use of Fig. 14.7 and (14-34)-(14-37).

From there

$$\mu'_x(t) = \frac{d\mu_x(t)}{dt} = \frac{d\lambda}{dt} = \lambda, \quad a\ constant \quad (14-14)$$

and

$$R_{xx'}(t_1, t_2) = \frac{\partial R_{xx}(t_1, t_2)}{\partial t_2} = \begin{cases} \lambda^2 t_1 & t_1 \leq t_2 \\ \lambda^2 t_1 + \lambda & t_1 > t_2 \end{cases}$$

$$= \lambda^2 t_1 + \lambda \ U(t_1 - t_2) \quad (14-14)$$

and

$$R_{xx'}(t_1, t_2) = \frac{\partial R_{xx'}(t_1, t_2)}{\partial t_1} = \lambda^2 + \lambda \ \delta(t_1 - t_2). \quad (14-16)$$
From (14-14) and (14-16) it follows that $X'(t)$ is a wide sense stationary process. Thus nonstationary inputs to linear systems can lead to wide sense stationary outputs, an interesting observation.

- **Sum of Poisson Processes:**
  If $X_1(t)$ and $X_2(t)$ represent two independent Poisson processes, then their sum $X_1(t) + X_2(t)$ is also a Poisson process with parameter $(\lambda_1 + \lambda_2)t$. (Follows from (6-86), Text and the definition of the Poisson process in (i) and (ii)).

- **Random selection of Poisson Points:**
  Let $t_1, t_2, \cdots, t_i, \cdots$ represent random arrival points associated with a Poisson process $X(t)$ with parameter $\lambda t$, and associated with each arrival point, define an independent Bernoulli random variable $N_i$, where
  \[
  P(N_i = 1) = p, \quad P(N_i = 0) = q = 1 - p. \tag{14-17}
  \]
Define the processes

\[
Y(t) = \sum_{i=1}^{X(t)} N_i \quad ; \quad Z(t) = \sum_{i=1}^{X(t)} (1 - N_i) = X(t) - Y(t)
\] (14-18)

we claim that both \(Y(t)\) and \(Z(t)\) are independent Poisson processes with parameters \(\lambda pt\) and \(\lambda qt\) respectively.

**Proof:**

\[
Y(t) = \sum_{n=k}^{\infty} P\{Y(t) = k \mid X(t) = n\} P\{X(t) = n\}.
\] (14-19)

But given \(X(t) = n\), we have \(Y(t) = \sum_{i=1}^{n} N_i \sim B(n, p)\) so that

\[
P\{Y(t) = k \mid X(t) = n\} = \binom{n}{k} p^k q^{n-k}, \quad 0 \leq k \leq n,
\] (14-20)

and

\[
P\{X(t) = n\} = e^{-\lambda t} \frac{(\lambda t)^n}{n!}.
\] (14-21)

Substituting (14-20)-(14-21) into (14-19) we get
\[ P\{Y(t) = k\} = e^{-\lambda t} \sum_{n=k}^{\infty} \frac{n!}{(n-k)!k!} p^k q^{n-k} \frac{(\lambda t)^n}{n!} = \frac{p^k e^{-\lambda t}}{k!} (\lambda t)^k \sum_{n=k}^{\infty} \frac{(q\lambda t)^{n-k}}{(n-k)!} e^{q\lambda t} \]

\[ = (\lambda pt)^k \frac{e^{-(1-q)\lambda t}}{k!} = e^{-\lambda pt} \frac{(\lambda pt)^k}{k!}, \quad k = 0, 1, 2, \ldots \]

\[ \sim P(\lambda pt). \quad (14-22) \]

More generally,

\[ P\{Y(t) = k, Z(t) = m\} = P\{Y(t) = k, X(t) - Y(t) = m\} \]

\[ = P\{Y(t) = k, X(t) = k + m\} \]

\[ = P\{Y(t) = k \mid X(t) = k + m\}P\{X(t) = k + m\} \]

\[ = \binom{k+m}{k} p^k q^m \cdot e^{-\lambda t} \frac{(\lambda t)^{k+m}}{(k+m)!} \frac{e^{-\lambda pt} (\lambda pt)^k}{k!} \frac{e^{-\lambda qt} (\lambda qt)^m}{m!} \]

\[ = P\{Y(t) = k\}P\{Z(t) = m\}, \quad (14-23) \]
which completes the proof.

Notice that $Y(t)$ and $Z(t)$ are generated as a result of random Bernoulli selections from the original Poisson process $X(t)$ (Fig. 14.5), where each arrival gets tossed over to either $Y(t)$ with probability $p$ or to $Z(t)$ with probability $q$. Each such sub-arrival stream is also a Poisson process. Thus random selection of Poisson points preserve the Poisson nature of the resulting processes. However, as we shall see deterministic selection from a Poisson process destroys the Poisson property for the resulting processes.

Fig. 14.5
**Inter-arrival Distribution for Poisson Processes**

Let $\tau_1$ denote the time interval (delay) to the first arrival from any fixed point $t_0$. To determine the probability distribution of the random variable $\tau_1$, we argue as follows: Observe that the event "$\tau_1 > t$" is the same as "$n(t_0, t_0+t) = 0$", or the complement event "$\tau_1 \leq t$" is the same as the event "$n(t_0, t_0+t) > 0$". Hence the distribution function of $\tau_1$ is given by

$$F_{\tau_1}(t) \triangleq P\{\tau_1 \leq t\} = P\{X(t) > 0\} = P\{n(t_0, t_0+t) > 0\}$$

$$= 1 - P\{n(t_0, t_0+t) = 0\} = 1 - e^{-\lambda t}$$  \hspace{1cm} (14-24)

(use (14-5)), and hence its derivative gives the probability density function for $\tau_1$ to be

$$f_{\tau_1}(t) = \frac{dF_{\tau_1}(t)}{dt} = \lambda e^{-\lambda t}, \quad t \geq 0$$  \hspace{1cm} (14-25)

i.e., $\tau_1$ is an exponential random variable with parameter $\lambda$ so that $E(\tau_1) = 1/\lambda$. 

Fig. 14.6
Similarly, let \( t_n \) represent the \( n^{th} \) random arrival point for a Poisson process. Then
\[
F_{t_n}(t) \triangleq P\{t_n \leq t\} = P\{X(t) \geq n\}
\]
\[
= 1 - P\{X(t) < n\} = 1 - \sum_{k=0}^{n-1} \frac{(\lambda t)^k}{k!} e^{-\lambda t}
\]
and hence
\[
f_{t_n}(x) = \frac{dF_{t_n}(x)}{dx} = -\sum_{k=1}^{n-1} \frac{\lambda(\lambda x)^{k-1}}{(k-1)!} e^{-\lambda x} + \sum_{k=0}^{n-1} \frac{\lambda(\lambda x)^k}{k!} e^{-\lambda x}
\]
\[
= \frac{\lambda^n x^{n-1}}{(n-1)!} e^{-\lambda x}, \quad x \geq 0
\]
(14-27)

which represents a gamma density function. i.e., the waiting time to the \( n^{th} \) Poisson arrival instant has a gamma distribution. Moreover
\[
t_n = \sum_{i=1}^{n} \tau_i
\]
where $\tau_i$ is the random inter-arrival duration between the $(i - 1)^{th}$ and $i^{th}$ events. Notice that $\tau_i$s are independent, identically distributed random variables. Hence using their characteristic functions, it follows that all inter-arrival durations of a Poisson process are independent exponential random variables with common parameter $\lambda$. i.e.,

$$f_{\tau_i}(t) = \lambda e^{-\lambda t}, \quad t \geq 0.$$  \hspace{1cm} (14-28)

Alternatively, from (14-24)-(14-25), we have $\tau_1$ is an exponential random variable. By repeating that argument after shifting $t_0$ to the new point $t_1$ in Fig. 14.6, we conclude that $\tau_2$ is an exponential random variable. Thus the sequence $\tau_1, \tau_2, \ldots, \tau_n, \ldots$ are independent exponential random variables with common p.d.f as in (14-25).

Thus if we systematically tag every $m^{th}$ outcome of a Poisson process $X(t)$ with parameter $\lambda t$ to generate a new process $e(t)$, then the inter-arrival time between any two events of $e(t)$ is a gamma random variable.
Notice that

\[ E[e(t)] = m/\lambda, \text{ and if } \lambda = m\mu, \text{ then } E[e(t)] = 1/\mu. \]

The inter-arrival time of \( e(t) \) in that case represents an Erlang-m random variable, and \( e(t) \) an Erlang-m process (see (10-90), Text). In summary, if Poisson arrivals are randomly redirected to form new queues, then each such queue generates a new Poisson process (Fig. 14.5). However if the arrivals are systematically redirected (1\(^{st}\) arrival to 1\(^{st}\) counter, 2\(^{nd}\) arrival to 2\(^{nd}\) counter, \( \cdots \), \( m^{th} \) to \( m^{th} \), (\( m+1 \)^{st} arrival to 1\(^{st}\) counter, \( \cdots \)), then the new subqueues form Erlang-m processes.

Interestingly, we can also derive the key Poisson properties (14-5) and (14-25) by starting from a simple axiomatic approach as shown below: