2. Independence and Bernoulli Trials

Independence: Events A and B are independent if

$$P(AB) = P(A)P(B). \tag{2-1}$$

- It is easy to show that A, B independent implies $\overline{A}, B$; $A, \overline{B}$; $\overline{A}, \overline{B}$ are all independent pairs. For example, $B = (A \cup \overline{A})B = AB \cup \overline{A}B$ and $AB \cap \overline{A}B = \phi$, so that

$$P(B) = P(AB \cup \overline{A}B) = P(AB) + P(\overline{A}B) = P(A)P(B) + P(\overline{A}B)$$

or

$$P(\overline{A}B) = P(B) - P(A)P(B) = (1 - P(A))P(B) = P(\overline{A})P(B),$$

i.e., $\overline{A}$ and B are independent events.

As an application, let A_p and A_q represent the events

$$A_p = \text{"the prime } p \text{ divides the number } N\text{"}$$

and

$$A_q = \text{"the prime } q \text{ divides the number } N\text{"}.$$

Then from (1-4)

$$P\{A_p\} = \frac{1}{p}, \qquad P\{A_q\} = \frac{1}{q}$$

Also

$$P\{A_p \cap A_q\} = P\{\text{"} pq \text{ divides } N\text{"}\} = \frac{1}{pq} = P\{A_p\}\, P\{A_q\} \tag{2-2}$$

Hence it follows that A_p and A_q are independent events!

- If $P(A) = 0$, then since the event $AB \subset A$ always, we have

$$P(AB) \le P(A) = 0 \Rightarrow P(AB) = 0,$$

and (2-1) is always satisfied. Thus the event of zero probability is independent of every other event!

- Independent events obviously cannot be mutually exclusive, since $P(A) > 0,\ P(B) > 0$ and A, B independent implies $P(AB) > 0.$ Thus if A and B are independent, the event AB cannot be the null set.

- More generally, a family of events $\{A_i\}$ are said to be independent, if for every finite sub collection $A_{i_1}, A_{i_2}, \cdots, A_{i_n},$ we have

$$P\left(\bigcap_{k=1}^{n} A_{i_k} \right) = \prod_{k=1}^{n} P(A_{i_k}). \qquad \text{(2-3)}$$

- Let

$$A = A_1 \cup A_2 \cup A_3 \cup \cdots \cup A_n, \tag{2-4}$$

a union of n independent events. Then by De-Morgan's law

$$\overline{A} = \overline{A_1}\,\overline{A_2} \cdots \overline{A_n}$$

and using their independence

$$P(\overline{A}) = P(\overline{A_1}\,\overline{A_2} \cdots \overline{A_n}) = \prod_{i=1}^{n} P(\overline{A_i}) = \prod_{i=1}^{n} (1 - P(A_i)). \tag{2-5}$$

Thus for any A as in (2-4)

$$P(A) = 1 - P(\overline{A}) = 1 - \prod_{i=1}^{n} (1 - P(A_i)), \tag{2-6}$$

a useful result.

Example 2.1: Three switches connected in parallel operate independently. Each switch remains closed with probability p. (a) Find the probability of receiving an input signal at the output. (b) Find the probability that switch S_1 is open given that an input signal is received at the output.

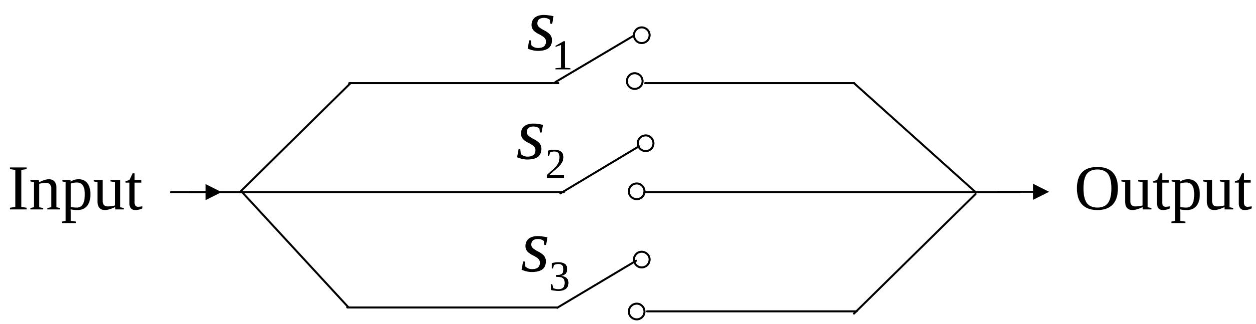

Fig.2.1

Solution: **a.** Let A_i = "Switch S_i is closed". Then $P(A_i) = p$, $i = 1 \to 3$. Since switches operate independently, we have

$$P(A_i A_j) = P(A_i)P(A_j); \; P(A_1 A_2 A_3) = P(A_1)P(A_2)P(A_3).$$

Let R = "input signal is received at the output". For the event R to occur either switch 1 or switch 2 or switch 3 must remain closed, i.e.,

$$R = A_1 \cup A_2 \cup A_3. \tag{2-7}$$

Using (2-3) - (2-6),

$$P(R) = P(A_1 \cup A_2 \cup A_3) = 1 - (1-p)^3 = 3p - 3p^2 + p^3. \tag{2-8}$$

We can also derive (2-8) in a different manner. Since any event and its compliment form a trivial partition, we can always write

$$P(R) = P(R \mid A_1)P(A_1) + P(R \mid \overline{A}_1)P(\overline{A}_1). \tag{2-9}$$

But $P(R \mid A_1) = 1,$ and $P(R \mid \overline{A}_1) = P(A_2 \cup A_3) = 2p - p^2$

and using these in (2-9) we obtain

$$P(R) = p + (2p - p^2)(1-p) = 3p - 3p^2 + p^3, \tag{2-10}$$

which agrees with (2-8).

Note that the events A_1, A_2, A_3 do not form a partition, since they are not mutually exclusive. Obviously any two or all three switches can be closed (or open) simultaneously. Moreover, $P(A_1) + P(A_2) + P(A_3) \neq 1.$

b. We need $P(\overline{A}_1 \mid R).$ From Bayes' theorem

$$P(\overline{A}_1 \mid R) = \frac{P(R \mid \overline{A}_1)P(\overline{A}_1)}{P(R)} = \frac{(2p - p^2)(1-p)}{3p - 3p^2 + p^3} = \frac{2 - 2p + p^2}{3p - 3p^2 + p^3}. \quad (2\text{-}11)$$

Because of the symmetry of the switches, we also have

$$P(\overline{A}_1 \mid R) = P(\overline{A}_2 \mid R) = P(\overline{A}_3 \mid R).$$

Repeated Trials

Consider two independent experiments with associated probability models (Ω_1, F_1, P_1) and (Ω_2, F_2, P_2). Let $\xi \in \Omega_1$, $\eta \in \Omega_2$ represent elementary events. A joint performance of the two experiments produces an elementary events $\omega = (\xi, \eta)$. How to characterize an appropriate probability to this "combined event" ? Towards this, consider the Cartesian product space $\Omega = \Omega_1 \times \Omega_2$ generated from Ω_1 and Ω_2 such that if $\xi \in \Omega_1$ and $\eta \in \Omega_2$, then every ω in Ω is an ordered pair of the form $\omega = (\xi, \eta)$. To arrive at a probability model we need to define the combined trio (Ω, F, P).

Suppose $A \in F_1$ and $B \in F_2$. Then $A \times B$ is the set of all pairs (ξ, η), where $\xi \in A$ and $\eta \in B$. Any such subset of Ω appears to be a legitimate event for the combined experiment. Let F denote the field composed of all such subsets $A \times B$ together with their unions and compliments. In this combined experiment, the probabilities of the events $A \times \Omega_2$ and $\Omega_1 \times B$ are such that

$$P(A \times \Omega_2) = P_1(A), \quad P(\Omega_1 \times B) = P_2(B). \qquad \text{(2-12)}$$

Moreover, the events $A \times \Omega_2$ and $\Omega_1 \times B$ are independent for any $A \in F_1$ and $B \in F_2$. Since

$$(A \times \Omega_2) \cap (\Omega_1 \times B) = A \times B, \qquad \text{(2-13)}$$

we conclude using (2-12) that

$$P(A \times B) = P(A \times \Omega_2) \cdot P(\Omega_1 \times B) = P_1(A) P_2(B) \qquad \text{(2-14)}$$

for all $A \in F_1$ and $B \in F_2$. The assignment in (2-14) extends to a unique probability measure $P (\equiv P_1 \times P_2)$ on the sets in F and defines the combined trio (Ω, F, P).

Generalization: Given n experiments $\Omega_1, \Omega_2, \cdots, \Omega_n,$ and their associated F_i and $P_i,\ i = 1 \to n$, let

$$\Omega = \Omega_1 \times \Omega_2 \times \cdots \times \Omega_n \qquad \text{(2-15)}$$

represent their Cartesian product whose elementary events are the ordered n-tuples $\xi_1, \xi_2, \cdots, \xi_n,$ where $\xi_i \in \Omega_i$. Events in this combined space are of the form

$$A_1 \times A_2 \times \cdots \times A_n \qquad \text{(2-16)}$$

where $A_i \in F_i,$ and their unions an intersections.

If all these n experiments are independent, and $P_i(A_i)$ is the probability of the event A_i in F_i then as before

$$P(A_1 \times A_2 \times \cdots \times A_n) = P_1(A_1)P_2(A_2)\cdots P_n(A_n). \qquad (2\text{-}17)$$

Example 2.2: An event A has probability p of occurring in a single trial. Find the probability that A occurs exactly k times, $k \leq n$ in n trials.

Solution: Let (Ω, F, P) be the probability model for a single trial. The outcome of n experiments is an n-tuple

$$\omega = \{\xi_1, \xi_2, \cdots, \xi_n\} \in \Omega_0, \qquad (2\text{-}18)$$

where every $\xi_i \in \Omega$ and $\Omega_0 = \Omega \times \Omega \times \cdots \times \Omega$ as in (2-15). The event A occurs at trial # i , if $\xi_i \in A$. Suppose A occurs exactly k times in ω.

Then k of the ξ_i belong to A, say $\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_k}$, and the remaining $n-k$ are contained in its compliment in $\overline{A}$. Using (2-17), the probability of occurrence of such an ω is given by

$$P_0(\omega) = P(\{\xi_{i_1}, \xi_{i_2}, \cdots, \xi_{i_k}, \cdots, \xi_{i_n}\}) = P(\{\xi_{i_1}\})P(\{\xi_{i_2}\})\cdots P(\{\xi_{i_k}\})\cdots P(\{\xi_{i_n}\})$$

$$= \underbrace{P(A)P(A)\cdots P(A)}_{k}\ \underbrace{P(\overline{A})P(\overline{A})\cdots P(\overline{A})}_{n-k} = p^k q^{n-k}. \qquad \text{(2-19)}$$

However the k occurrences of A can occur in any particular location inside ω. Let $\omega_1, \omega_2, \cdots, \omega_N$ represent all such events in which A occurs exactly k times. Then

$$\text{"}A \text{ occurs exactly } k \text{ times in } n \text{ trials"} = \omega_1 \cup \omega_2 \cup \cdots \cup \omega_N. \qquad \text{(2-20)}$$

But, all these ω_is are mutually exclusive, and equiprobable.

Thus

$$P(\text{"}A \text{ occurs exactly } k \text{ times in } n \text{ trials"})$$

$$= \sum_{i=1}^{N} P_0(\omega_i) = N P_0(\omega) = N p^k q^{n-k}, \qquad \text{(2-21)}$$

where we have used (2-19). Recall that, starting with n possible choices, the first object can be chosen n different ways, and for every such choice the second one in $(n-1)$ ways, … and the kth one $(n-k+1)$ ways, and this gives the total choices for k objects out of n to be $n\,(n-1)\cdots(n-k+1)$. But, this includes the $k!$ choices among the k objects that are indistinguishable for identical objects. As a result

$$N = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{(n-k)!\,k!} \triangleq \binom{n}{k} \qquad \text{(2-22)}$$

represents the number of combinations, or choices of n identical objects taken k at a time. Using (2-22) in (2-21), we get

$$P_n(k) = P(\text{"}A \text{ occurs exactly } k \text{ times in } n \text{ trials"})$$

$$= \binom{n}{k} p^k q^{n-k}, \quad k = 0,1,2,\cdots,n, \tag{2-23}$$

a formula, due to Bernoulli.

Independent repeated experiments of this nature, where the outcome is either a "success" $(= A)$ or a "failure" $(= \overline{A})$ are characterized as Bernoulli trials, and the probability of k successes in n trials is given by (2-23), where p represents the probability of "success" in any one trial.

Example 2.3: Toss a coin n times. Obtain the probability of getting k heads in n trials ?

Solution: We may identify "head" with "success" (A) and let $p = P(H)$. In that case (2-23) gives the desired probability.

Example 2.4: Consider rolling a fair die eight times. Find the probability that either 3 or 4 shows up five times ?

Solution: In this case we can identify

$$\text{"success"} = A = \{ \text{ either } 3 \text{ or } 4 \} = \{f_3\} \cup \{f_4\}.$$

Thus

$$P(A) = P(f_3) + P(f_4) = \frac{1}{6} + \frac{1}{6} = \frac{1}{3},$$

and the desired probability is given by (2-23) with $n = 8,\ k = 5$ and $p = 1/3.$ Notice that this is similar to a "biased coin" problem.

Bernoulli trial: consists of repeated independent and identical experiments each of which has only two outcomes A or $\overline{A}$ with $P(A) = p,$ and $P(\overline{A}) = q$. The probability of exactly k occurrences of A in n such trials is given by (2-23).

Let

$$X_k = \text{"exactly } k \text{ occurrences in } n \text{ trials"}. \qquad (2\text{-}24)$$

Since the number of occurrences of A in n trials must be an integer $k = 0, 1, 2, \cdots, n,$ either X_0 or X_1 or X_2 or $\cdots$ or X_n must occur in such an experiment. Thus

$$P(X_0 \cup X_1 \cup \cdots \cup X_n) = 1. \qquad (2\text{-}25)$$

But $X_i,\ X_j$ are mutually exclusive. Thus

$$P(X_0 \cup X_1 \cup \cdots \cup X_n) = \sum_{k=0}^{n} P(X_k) = \sum_{k=0}^{n} \binom{n}{k} p^k q^{n-k}. \quad (2\text{-}26)$$

From the relation

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}, \qquad (2\text{-}27)$$

(2-26) equals $(p+q)^n = 1,$ and it agrees with (2-25).

For a given *n* and *p* what is the most likely value of *k* ? From Fig.2.2, the most probable value of *k* is that number which maximizes $P_n(k)$ in (2-23). To obtain this value, consider the ratio

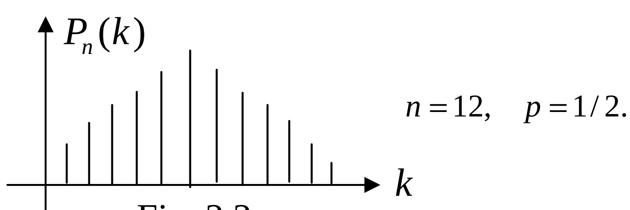

Fig. 2.2

$$\frac{P_n(k-1)}{P_n(k)} = \frac{n!\, p^{k-1} q^{n-k+1}}{(n-k+1)!\,(k-1)!} \frac{(n-k)!\, k!}{n!\, p^k q^{n-k}} = \frac{k}{n-k+1} \frac{q}{p}. \qquad (2\text{-}28)$$

Thus $P_n(k) \geq P_n(k-1)$, if $k(1-p) \leq (n-k+1)p$ or $k \leq (n+1)p$. Thus $P_n(k)$ as a function of k increases until

$$k = (n+1)p \qquad (2\text{-}29)$$

if it is an integer, or the largest integer $k_{\max}$ less than $(n+1)p$, and (2-29) represents the most likely number of successes (or heads) in n trials.

Example 2.5: In a Bernoulli experiment with n trials, find the probability that the number of occurrences of A is between k_1 and k_2.

Solution: With $X_i,\ i = 0,1,2,\cdots,n,$ as defined in (2-24), clearly they are mutually exclusive events. Thus

$$P(\text{" Occurrences of } A \text{ is between } k_1 \text{ and } k_2\text{"})$$

$$= P(X_{k_1} \cup X_{k_1+1} \cup \cdots \cup X_{k_2}) = \sum_{k=k_1}^{k_2} P(X_k) = \sum_{k=k_1}^{k_2} \binom{n}{k} p^k q^{n-k}. \quad \text{(2-30)}$$

Example 2.6: Suppose 5,000 components are ordered. The probability that a part is defective equals 0.1. What is the probability that the total number of defective parts does not exceed 400 ?

Solution: Let

$$Y_k = \text{"} k \text{ parts are defective among 5,000 components"}.$$

Using (2-30), the desired probability is given by

$$P(Y_0 \cup Y_1 \cup \cdots \cup Y_{400}) = \sum_{k=0}^{400} P(Y_k)$$
$$= \sum_{k=0}^{400} \binom{5000}{k} (0.1)^k (0.9)^{5000-k}. \qquad (2\text{-}31)$$