5. Functions of a Random Variable

Let X be a r.v defined on the model $(\Omega, F, P),$ and suppose $g(x)$ is a function of the variable x. Define

$$Y = g(X). \tag{5-1}$$

Is Y necessarily a r.v? If so what is its cdf $F_Y(y),$ and pdf $f_Y(y)?$

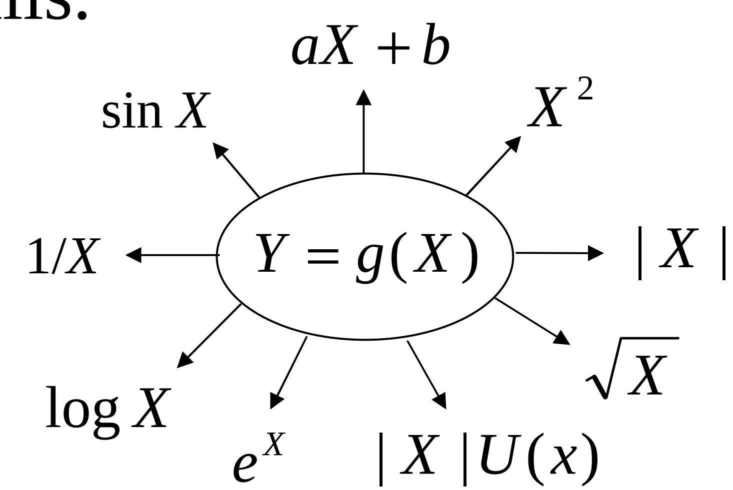
Clearly if Y is a r.v, then for every Borel set B, the set of ξ for which $Y(\xi) \in B$ must belong to F. Given that X is a r.v, this is assured if $g^{-1}(B)$ is also a Borel set, i.e., if $g(x)$ is a Borel function. In that case if X is a r.v, so is Y, and for every Borel set B

$$P(Y \in B) = P(X \in g^{-1}(B)). \tag{5-2}$$

In particular

$$F_Y(y) = P(Y(\xi) \le y) = P\big(g(X(\xi)) \le y\big) = P\Big(X(\xi) \le g^{-1}(-\infty, y]\Big). \quad \text{(5-3)}$$

Thus the distribution function as well of the density function of Y can be determined in terms of that of X. To obtain the distribution function of Y, we must determine the Borel set on the x-axis such that $X(\xi) \le g^{-1}(y)$ for every given y, and the probability of that set. At this point, we shall consider some of the following functions to illustrate the technical details.



Example 5.1: $Y = aX + b$ (5-4)

Solution: Suppose $a > 0$.

$$F_Y(y) = P(Y(\xi) \le y) = P(aX(\xi) + b \le y) = P\left(X(\xi) \le \frac{y-b}{a} \right) = F_X\left(\frac{y-b}{a} \right). \quad (5\text{-}5)$$

and

$$f_Y(y) = \frac{1}{a} f_X\left(\frac{y-b}{a} \right). \quad (5\text{-}6)$$

On the other hand if $a < 0$, then

$$F_Y(y) = P(Y(\xi) \le y) = P(aX(\xi) + b \le y) = P\left(X(\xi) > \frac{y-b}{a} \right)$$

$$= 1 - F_X\left(\frac{y-b}{a} \right), \quad (5\text{-}7)$$

and hence

$$f_Y(y) = -\frac{1}{a} f_X\left(\frac{y-b}{a} \right). \quad (5\text{-}8)$$

From (5-6) and (5-8), we obtain (for all a)

$$f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right). \tag{5-9}$$

Example 5.2: $Y = X^2.$ (5-10)

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\big(X^2(\xi) \le y\big). \tag{5-11}$$

If $y < 0,$ then the event $\left\{X^2(\xi) \le y\right\} = \phi,$ and hence

$$F_Y(y) = 0, \quad y < 0. \tag{5-12}$$

For $y > 0,$ from Fig. 5.1, the event $\{Y(\xi) \le y\} = \{X^2(\xi) \le y\}$ is equivalent to $\{x_1 < X(\xi) \le x_2\}.$

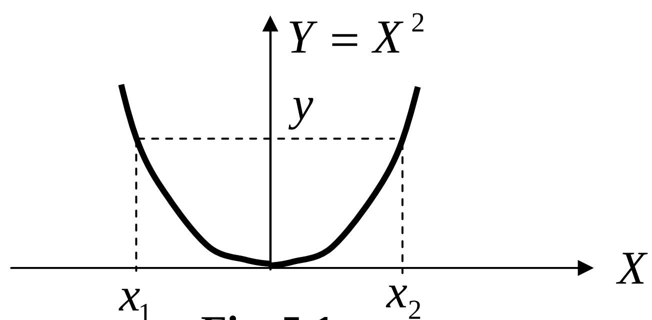


Fig. 5.1

Hence

$$F_Y(y) = P\left(x_1 < X(\xi) \le x_2\right) = F_X(x_2) - F_X(x_1)$$
$$= F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y > 0. \qquad \text{(5-13)}$$

By direct differentiation, we get

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right), & y > 0, \\ 0, & \text{otherwise} \end{cases}. \qquad \text{(5-14)}$$

If $f_X(x)$ represents an even function, then (5-14) reduces to

$$f_Y(y) = \frac{1}{\sqrt{y}} f_X\left(\sqrt{y}\right) U(y). \qquad \text{(5-15)}$$

In particular if $X \sim N(0,1)$, so that

$$f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad \text{(5-16)}$$

and substituting this into (5-14) or (5-15), we obtain the p.d.f of $Y = X^2$ to be

$$f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} U(y). \tag{5-17}$$

On comparing this with (3-36), we notice that (5-17) represents a Chi-square r.v with $n = 1$, since $\Gamma(1/2) = \sqrt{\pi}.$ Thus, if X is a Gaussian r.v with $\mu = 0,$ then $Y = X^2$ represents a Chi-square r.v with one degree of freedom ($n = 1$).

Example 5.3: Let

$$Y = g(X) = \begin{cases} X - c, & X > c, \\ 0, & -c < X \leq c, \\ X + c, & X \leq -c. \end{cases}$$

In this case

$$P(Y=0)=P(-c<X(\xi)\le c)=F_X(c)-F_X(-c). \tag{5-18}$$

For $y>0,$ we have $x>c,$ and $Y(\xi)=X(\xi)-c$ so that

$$\begin{aligned} F_Y(y) &= P\big(Y(\xi)\le y\big)=P(X(\xi)-c\le y) \\ &= P\big(X(\xi)\le y+c\big)=F_X(y+c), \quad y>0. \end{aligned} \tag{5-19}$$

Similarly $y<0,$ if $x<-c,$ and $Y(\xi)=X(\xi)+c$ so that

$$\begin{aligned} F_Y(y) &= P\big(Y(\xi)\le y\big)=P(X(\xi)+c\le y) \\ &= P\big(X(\xi)\le y-c\big)=F_X(y-c), \quad y<0. \end{aligned} \tag{5-20}$$

Thus

$$f_Y(y)=\begin{cases} f_X(y+c), & y>0, \\ [F_X(c)-F_X(-c)]\delta(y), & \\ f_X(y-c), & y<0. \end{cases} \tag{5-21}$$

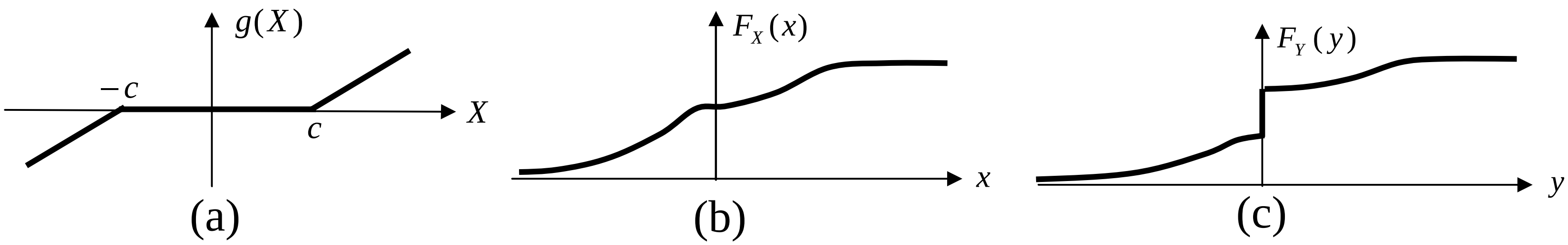


Fig. 5.2

Example 5.4: Half-wave rectifier

$$Y = g(X); \quad g(x) = \begin{cases} x, & x > 0, \\ 0, & x \le 0. \end{cases} \qquad (5\text{-}22)$$

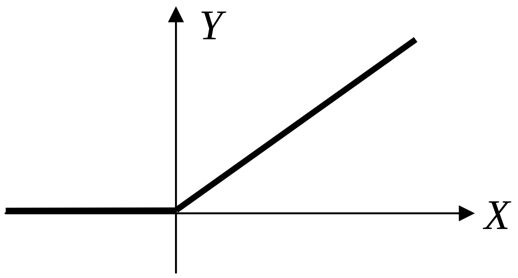


Fig. 5.3

In this case

$$P(Y = 0) = P(X(\xi) \le 0) = F_X(0). \qquad (5\text{-}23)$$

and for $y > 0,$ since $Y = X$,

$$F_Y(y) = P\big(Y(\xi) \le y\big) = P\big(X(\xi) \le y\big) = F_X(y). \qquad (5\text{-}24)$$

Thus

$$f_Y(y) = \begin{cases} f_X(y), & y > 0, \\ F_X(0)\delta(y) & y = 0, \\ 0, & y < 0, \end{cases} \quad = \quad f_X(y)U(y) + F_X(0)\delta(y). \qquad (5\text{-}25)$$

Note: As a general approach, given $Y = g(X)$, first sketch the graph $y = g(x)$, and determine the range space of y. Suppose $a < y < b$ is the range space of $y = g(x)$. Then clearly for $y < a$, $F_Y(y) = 0$, and for $y > b$, $F_Y(y) = 1$, so that $F_Y(y)$ can be nonzero only in $a < y < b$. Next, determine whether there are discontinuities in the range space of y. If so evaluate $P(Y(\xi) = y_i)$ at these discontinuities. In the continuous region of y, use the basic approach

$$F_Y(y) = P\big(g(X(\xi)) \le y\big)$$

and determine appropriate events in terms of the r.v X for every y. Finally, we must have $F_Y(y)$ for $-\infty < y < +\infty$, and obtain

$$f_Y(y) = \frac{dF_Y(y)}{dy} \quad \text{in} \quad a < y < b.$$

However, if $Y = g(X)$ is a continuous function, it is easy to establish a direct procedure to obtain $f_Y(y)$. A continuos function $g(x)$ with $g'(x)$ nonzero at all but a finite number of points, has only a finite number of maxima and minima, and it eventually becomes monotonic as $|x| \to \infty$. Consider a specific y on the y-axis, and a positive increment Δy as shown in Fig. 5.4

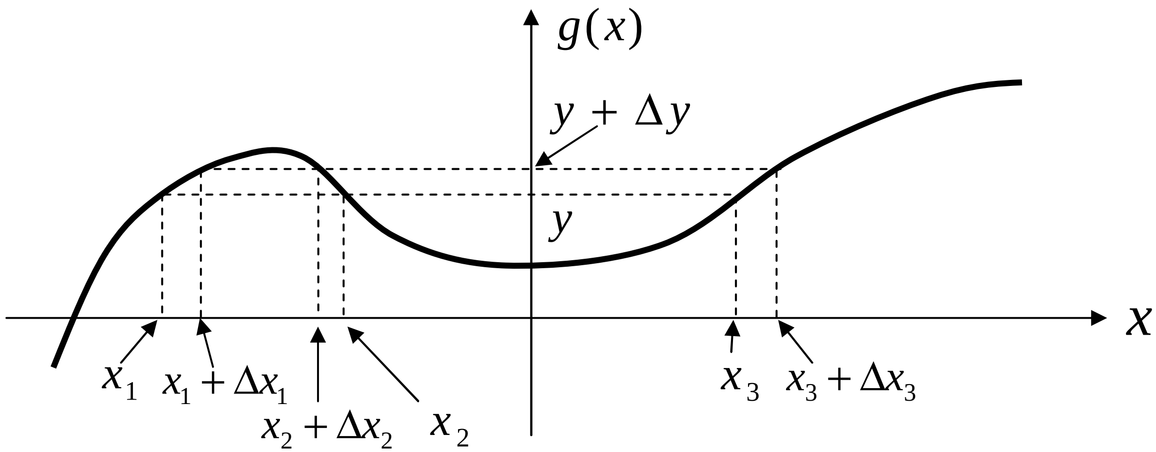


Fig. 5.4

$f_Y(y)$ for $Y = g(X)$, where $g(\cdot)$ is of continuous type.

Using (3-28) we can write

$$P\{y < Y(\xi) \le y + \Delta y\} = \int_y^{y+\Delta y} f_Y(u)\,du \approx f_Y(y) \cdot \Delta y. \qquad (5\text{-}26)$$

But the event $\{y < Y(\xi) \le y + \Delta y\}$ can be expressed in terms of $X(\xi)$ as well. To see this, referring back to Fig. 5.4, we notice that the equation $y = g(x)$ has three solutions x_1, x_2, x_3 (for the specific y chosen there). As a result when $\{y < Y(\xi) \le y + \Delta y\}$, the r.v X could be in any one of the three mutually exclusive intervals

$$\{x_1 < X(\xi) \le x_1 + \Delta x_1\},\ \{x_2 + \Delta x_2 < X(\xi) \le x_2\} \text{ or } \{x_3 < X(\xi) \le x_3 + \Delta x_3\}.$$

Hence the probability of the event in (5-26) is the sum of the probability of the above three events, i.e.,

$$P\{y < Y(\xi) \le y + \Delta y\} = P\{x_1 < X(\xi) \le x_1 + \Delta x_1\} + P\{x_2 + \Delta x_2 < X(\xi) \le x_2\} + P\{x_3 < X(\xi) \le x_3 + \Delta x_3\}. \qquad (5\text{-}27)$$

For small $\Delta y, \Delta x_i$, making use of the approximation in (5-26), we get

$$f_Y(y)\Delta y = f_X(x_1)\Delta x_1 + f_X(x_2)(-\Delta x_2) + f_X(x_3)\Delta x_3. \qquad (5\text{-}28)$$

In this case, $\Delta x_1 > 0$, $\Delta x_2 < 0$ and $\Delta x_3 > 0$, so that (5-28) can be rewritten as

$$f_Y(y) = \sum_i f_X(x_i)\frac{|\Delta x_i|}{\Delta y} = \sum_i \frac{1}{|\Delta y / \Delta x_i|} f_X(x_i) \qquad (5\text{-}29)$$

and as $\Delta y \to 0$, (5-29) can be expressed as

$$f_Y(y) = \sum_i \frac{1}{|dy/dx|_{x_i}} f_X(x_i) = \sum_i \frac{1}{|g'(x_i)|} f_X(x_i). \qquad (5\text{-}30)$$

The summation index *i* in (5-30) depends on *y*, and for every *y* the equation $y = g(x_i)$ must be solved to obtain the total number of solutions at every *y*, and the actual solutions $x_1, x_2, \cdots$ all in terms of *y*.

For example, if $Y = X^2$, then for all $y > 0$, $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$ represent the two solutions for each *y*. Notice that the solutions x_i are all in terms of *y* so that the right side of (5-30) is only a function of *y*. Referring back to the example $Y = X^2$ (Example 5.2) here for each $y > 0,$ there are two solutions given by $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}.$ ($f_Y(y) = 0$ for $y < 0$). Moreover

$$\frac{dy}{dx} = 2x \quad \text{so that} \quad \left|\frac{dy}{dx}\right|_{x = x_i} = 2\sqrt{y}$$

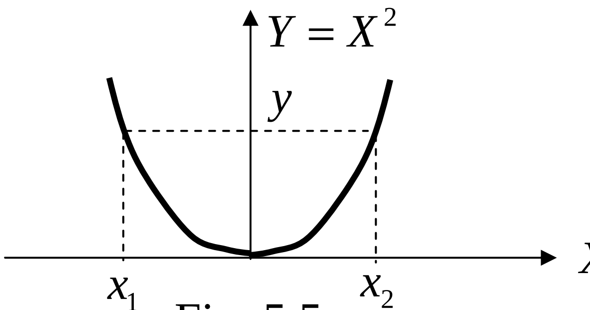


Fig. 5.5

and using (5-30) we get

$$f_Y(y) = \begin{cases} \dfrac{1}{2\sqrt{y}}\left(f_X(\sqrt{y}) + f_X(-\sqrt{y})\right), & y > 0, \\ 0, & \text{otherwise} \end{cases}, \qquad (5\text{-}31)$$

which agrees with (5-14).

Example 5.5: $Y = \dfrac{1}{X}.$ Find $f_Y(y).$ (5-32)

Solution: Here for every *y*, $x_1 = 1/y$ is the only solution, and

$$\frac{dy}{dx} = -\frac{1}{x^2} \quad \text{so that} \quad \left|\frac{dy}{dx}\right|_{x=x_1} = \frac{1}{1/y^2} = y^2,$$

and substituting this into (5-30), we obtain

$$f_Y(y) = \frac{1}{y^2} f_X\left(\frac{1}{y}\right). \tag{5-33}$$

In particular, suppose *X* is a Cauchy r.v as in (3-39) with parameter α so that

$$f_X(x) = \frac{\alpha/\pi}{\alpha^2 + x^2}, \quad -\infty < x < +\infty. \tag{5-34}$$

In that case from (5-33), $Y = 1/X$ has the p.d.f

$$f_Y(y) = \frac{1}{y^2}\frac{\alpha/\pi}{\alpha^2 + (1/y)^2} = \frac{(1/\alpha)/\pi}{(1/\alpha)^2 + y^2}, \quad -\infty < y < +\infty. \tag{5-35}$$

But (5-35) represents the p.d.f of a Cauchy r.v with parameter $1/\alpha.$ Thus if $X \sim C(\alpha),$ then $1/X \sim C(1/\alpha).$

Example 5.6: Suppose $f_X(x) = 2x/\pi^2, \quad 0 < x < \pi,$ and $Y = \sin X.$ Determine $f_Y(y).$

Solution: Since X has zero probability of falling outside the interval $(0, \pi),$ $y = \sin x$ has zero probability of falling outside the interval $(0,1).$ Clearly $f_Y(y) = 0$ outside this interval. For any $0 < y < 1,$ from Fig.5.6(b), the equation $y = \sin x$ has an infinite number of solutions $\cdots, x_1, x_2, x_3, \cdots,$ where $x_1 = \sin^{-1} y$ is the principal solution. Moreover, using the symmetry we also get $x_2 = \pi - x_1$ etc. Further,

$$\frac{dy}{dx} = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$$

so that

$$\left|\frac{dy}{dx}\right|_{x = x_i} = \sqrt{1 - y^2}.$$

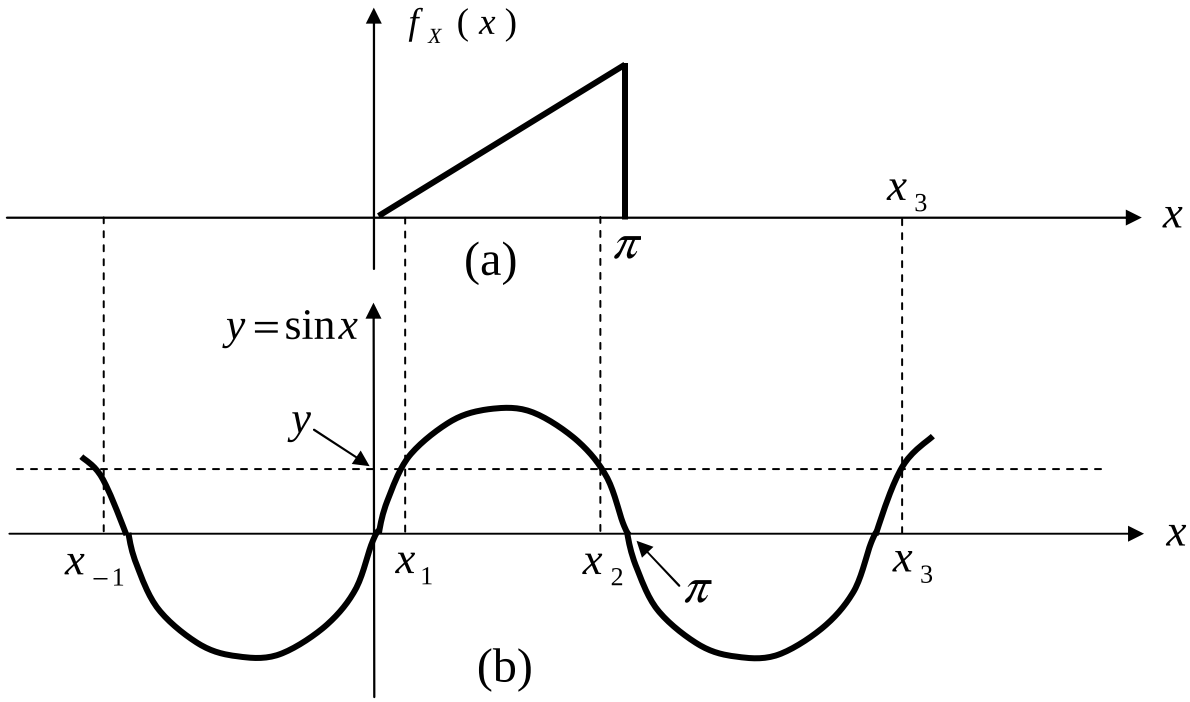


Fig. 5.6

Using this in (5-30), we obtain for $0 < y < 1,$

$$f_Y(y) = \sum_{\substack{i=-\infty \\ i\neq 0}}^{+\infty} \frac{1}{\sqrt{1-y^2}} f_X(x_i). \tag{5-36}$$

But from Fig. 5.6(a), in this case $f_X(x_{-1}) = f_X(x_3) = f_X(x_4) = \cdots = 0$ (Except for $f_X(x_1)$ and $f_X(x_2)$ the rest are all zeros).

Thus (Fig. 5.7)

$$f_Y(y) = \frac{1}{\sqrt{1-y^2}}\left(f_X(x_1) + f_X(x_2)\right) = \frac{1}{\sqrt{1-y^2}}\left(\frac{2x_1}{\pi^2} + \frac{2x_2}{\pi^2}\right)$$

$$= \frac{2(x_1+\pi-x_1)}{\pi^2\sqrt{1-y^2}} = \begin{cases} \dfrac{2}{\pi\sqrt{1-y^2}}, & 0 < y < 1, \\ 0, & \text{otherwise.} \end{cases} \quad (5\text{-}37)$$

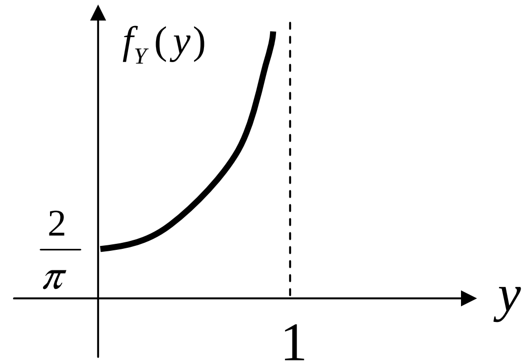


Fig. 5.7

Example 5.7: Let $Y = \tan X$ where $X \sim U(-\pi/2, \pi/2)$. Determine $f_Y(y)$.

Solution: As x moves from $(-\pi/2, \pi/2)$, y moves from $(-\infty, +\infty)$. From Fig.5.8(b), the function $Y = \tan X$ is one-to-one for $-\pi/2 < x < \pi/2$. For any y, $x_1 = \tan^{-1} y$ is the principal solution. Further

$$\frac{dy}{dx} = \frac{d \tan x}{dx} = \sec^2 x = 1 + \tan^2 x = 1 + y^2$$

so that using (5-30)

$$f_Y(y) = \frac{1}{|dy/dx|_{x=x_1}} f_X(x_1) = \frac{1/\pi}{1+y^2}, \qquad -\infty < y < +\infty, \quad (5\text{-}38)$$

which represents a Cauchy density function with parameter equal to unity (Fig. 5.9).

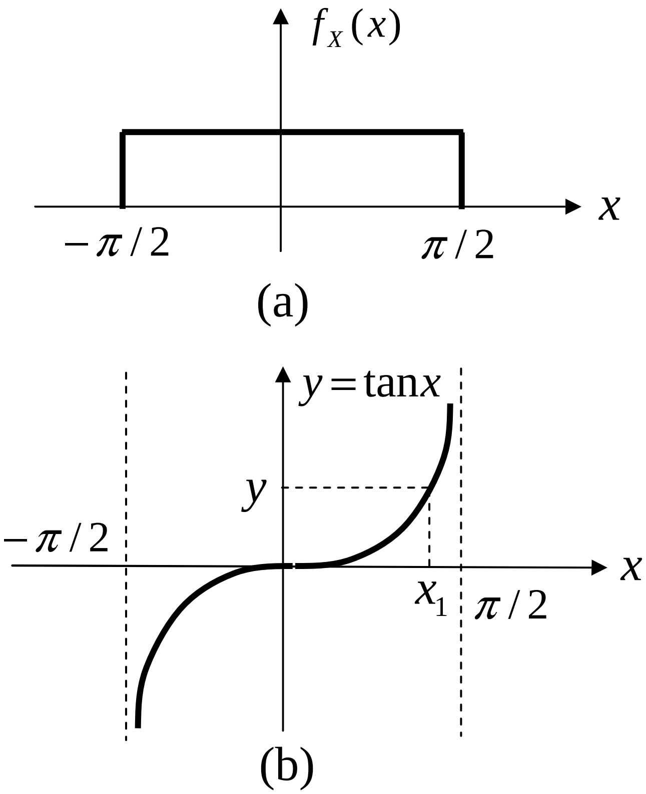


Fig. 5.8

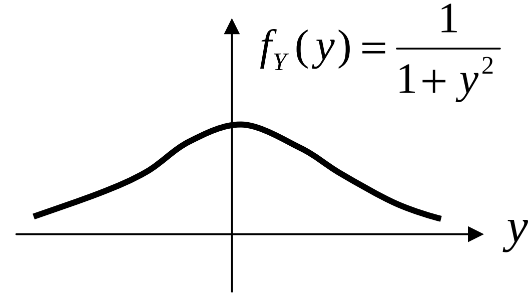


Fig. 5.9

Functions of a discrete-type r.v

Suppose X is a discrete-type r.v with

$$P(X = x_i) = p_i, \quad x = x_1, x_2, \cdots, x_i, \cdots \qquad (5\text{-}39)$$

and $Y = g(X)$. Clearly Y is also of discrete-type, and when $x = x_i, \quad y_i = g(x_i)$, and for those y_i

$$P(Y = y_i) = P(X = x_i) = p_i, \quad y = y_1, y_2, \cdots, y_i, \cdots \qquad (5\text{-}40)$$

Example 5.8: Suppose $X \sim P(\lambda)$, so that

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0,1,2,\cdots \qquad (5\text{-}41)$$

Define $Y = X^2 + 1$. Find the p.m.f of Y.

Solution: X takes the values $0,1,2, \cdots, k, \cdots$ so that Y only takes the value $1, 2, 5, \cdots, k^2 + 1, \cdots$ and

$$P(Y = k^2 + 1) = P(X = k)$$

so that for $j = k^2 + 1$

$$P(Y = j) = P\left(X = \sqrt{j-1}\right) = e^{-\lambda} \frac{\lambda^{\sqrt{j-1}}}{(\sqrt{j-1})!}, \quad j = 1, 2, 5, \cdots, k^2 + 1, \cdots. \quad (5\text{-}42)$$