5. Functions of a Random Variable

Let $X$ be a r.v defined on the model $(\Omega, F, P)$, and suppose $g(x)$ is a function of the variable $x$. Define

$$Y = g(X). \quad (5-1)$$

Is $Y$ necessarily a r.v? If so what is its cdf $F_Y(y)$, and pdf $f_Y(y)$?

Clearly if $Y$ is a r.v, then for every Borel set $B$, the set of $\xi$ for which $Y(\xi) \in B$ must belong to $F$. Given that $X$ is a r.v, this is assured if $g^{-1}(B)$ is also a Borel set, i.e., if $g(x)$ is a Borel function. In that case if $X$ is a r.v, so is $Y$, and for every Borel set $B$

$$P(Y \in B) = P(X \in g^{-1}(B)). \quad (5-2)$$
In particular

\[ F_Y(y) = P(Y(\xi) \leq y) = P(g(X(\xi)) \leq y) = P(X(\xi) \leq g^{-1}(-\infty, y]). \] (5-3)

Thus the distribution function as well of the density function of \( Y \) can be determined in terms of that of \( X \). To obtain the distribution function of \( Y \), we must determine the Borel set on the \( x \)-axis such that \( X(\xi) \leq g^{-1}(y) \) for every given \( y \), and the probability of that set. At this point, we shall consider some of the following functions to illustrate the technical details.

\[
\begin{align*}
Y &= g(X) \\
1/X &\quad \log X &\quad \sqrt{X} &\quad |X| &\quad |X|U(x) \\
aX + b &\quad X^2 &\quad \sin X &\quad e^X &\quad \text{ } \\
\end{align*}
\]
Example 5.1: \( Y = aX + b \) \hspace{1cm} (5-4)

Solution: Suppose \( a > 0 \).

\[
F_Y(y) = P(Y(\xi) \leq y) = P(aX(\xi) + b \leq y) = P\left( X(\xi) \leq \frac{y-b}{a} \right) = F_X\left( \frac{y-b}{a} \right). \hspace{1cm} (5-5)
\]

and

\[
f_Y(y) = \frac{1}{a} f_X \left( \frac{y-b}{a} \right). \hspace{1cm} (5-6)
\]

On the other hand if \( a < 0 \), then

\[
F_Y(y) = P(Y(\xi) \leq y) = P(aX(\xi) + b \leq y) = P\left( X(\xi) > \frac{y-b}{a} \right) = 1 - F_X\left( \frac{y-b}{a} \right), \hspace{1cm} (5-7)
\]

and hence

\[
f_Y(y) = -\frac{1}{a} f_X \left( \frac{y-b}{a} \right). \hspace{1cm} (5-8)
\]
From (5-6) and (5-8), we obtain (for all $a$)

$$f_y(y) = \frac{1}{|a|} f_x\left(\frac{y-b}{a}\right).$$  \hfill (5-9)

**Example 5.2:** $Y = X^2$.  \hfill (5-10)

$$F_Y(y) = P(Y(\xi) \leq y) = P\left(X^2(\xi) \leq y\right).$$  \hfill (5-11)

If $y < 0$, then the event $\{X^2(\xi) \leq y\} = \phi$, and hence

$$F_Y(y) = 0, \quad y < 0.$$  \hfill (5-12)

For $y > 0$, from Fig. 5.1, the event $\{Y(\xi) \leq y\} = \{X^2(\xi) \leq y\}$ is equivalent to $\{x_1 < X(\xi) \leq x_2\}$.  \hfill (5-12)
Hence

\[ F_Y(y) = P(x_1 < X(\xi) \leq x_2) = F_X(x_2) - F_X(x_1) \]
\[ = F_X(\sqrt{y}) - F_X(-\sqrt{y}), \quad y > 0. \] (5-13)

By direct differentiation, we get

\[
f_Y(y) = \begin{cases} 
\frac{1}{2\sqrt{y}} \left( f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right), & y > 0, \\
0, & \text{otherwise}. 
\end{cases} \] (5-14)

If \( f_X(x) \) represents an even function, then (5-14) reduces to

\[ f_Y(y) = \frac{1}{\sqrt{y}} f_X(\sqrt{y}) U(y). \] (5-15)

In particular if \( X \sim N(0,1) \), so that

\[ f_X(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \] (5-16)
and substituting this into (5-14) or (5-15), we obtain the p.d.f of \( Y = X^2 \) to be

\[
f_Y(y) = \frac{1}{\sqrt{2\pi y}} e^{-y/2} U(y).
\]  

(5-17)

On comparing this with (3-36), we notice that (5-17) represents a Chi-square r.v with \( n = 1 \), since \( \Gamma(1/2) = \sqrt{\pi} \).

Thus, if \( X \) is a Gaussian r.v with \( \mu = 0 \), then \( Y = X^2 \) represents a Chi-square r.v with one degree of freedom \( (n = 1) \).

**Example 5.3:** Let

\[
Y = g(X) = \begin{cases} 
X - c, & X > c, \\
0, & -c < X \leq c, \\
X + c, & X \leq -c.
\end{cases}
\]
In this case

\[ P(Y = 0) = P(-c < X(\xi) \leq c) = F_x(c) - F_x(-c). \] (5-18)

For \( y > 0 \), we have \( x > c \), and \( Y(\xi) = X(\xi) - c \) so that

\[ F_y(y) = P(Y(\xi) \leq y) = P(X(\xi) - c \leq y) \]
\[ = P(X(\xi) \leq y + c) = F_x(y + c), \quad y > 0. \] (5-19)

Similarly, \( y < 0 \), if \( x < -c \), and \( Y(\xi) = X(\xi) + c \) so that

\[ F_y(y) = P(Y(\xi) \leq y) = P(X(\xi) + c \leq y) \]
\[ = P(X(\xi) \leq y - c) = F_x(y - c), \quad y < 0. \] (5-20)

Thus

\[ f_y(y) = \begin{cases} 
  f_x(y + c), & y > 0, \\
  [F_x(c) - F_x(-c)]\delta(y), & \\
  f_x(y - c), & y < 0. 
\end{cases} \] (5-21)
Example 5.4: Half-wave rectifier

\[ Y = g(X); \quad g(x) = \begin{cases} x, & x > 0, \\ 0, & x \leq 0. \end{cases} \quad (5-22) \]

In this case,

\[ P(Y = 0) = P(X(\xi) \leq 0) = F_x(0). \quad (5-23) \]

and for \( y > 0 \), since \( Y = X \),

\[ F_y(y) = P(Y(\xi) \leq y) = P(X(\xi) \leq y) = F_x(y). \quad (5-24) \]

Thus

\[ f_y(y) = \begin{cases} f_x(y), & y > 0, \\ F_x(0)\delta(y) & y = 0, \\ 0, & y < 0, \end{cases} = f_x(y)U(y) + F_x(0)\delta(y). \quad (5-25) \]
Note: As a general approach, given \( Y = g(X) \), first sketch the graph \( y = g(x) \), and determine the range space of \( y \).
Suppose \( a < y < b \) is the range space of \( y = g(x) \).
Then clearly for \( y < a \), \( F_y(y) = 0 \), and for \( y > b \), \( F_y(y) = 1 \), so that \( F_y(y) \) can be nonzero only in \( a < y < b \). Next, determine whether there are discontinuities in the range space of \( y \). If so evaluate \( P(Y(\xi) = y_i) \) at these discontinuities. In the continuous region of \( y \), use the basic approach

\[
F_y(y) = P(g(X(\xi)) \leq y)
\]

and determine appropriate events in terms of the r.v \( X \) for every \( y \). Finally, we must have \( F_y(y) \) for \( -\infty < y < +\infty \), and obtain

\[
f_y(y) = \frac{dF_y(y)}{dy} \quad \text{in} \quad a < y < b.
\]
However, if $Y = g(X)$ is a continuous function, it is easy to establish a direct procedure to obtain $f_Y(y)$. A continuous function $g(x)$ with $g'(x)$ nonzero at all but a finite number of points, has only a finite number of maxima and minima, and it eventually becomes monotonic as $|x| \to \infty$. Consider a specific $y$ on the $y$-axis, and a positive increment $\Delta y$ as shown in Fig. 5.4

$$f_Y(y) \text{ for } Y = g(X), \text{ where } g(\cdot) \text{ is of continuous type.}$$
Using (3-28) we can write

\[ P\{y < Y(\xi) \leq y + \Delta y\} = \int_{y}^{y+\Delta y} f_y(u)du \approx f_y(y) \cdot \Delta y. \] (5-26)

But the event \( \{y < Y(\xi) \leq y + \Delta y\} \) can be expressed in terms of \( X(\xi) \) as well. To see this, referring back to Fig. 5.4, we notice that the equation \( y=g(x) \) has three solutions \( x_1, x_2, x_3 \) (for the specific \( y \) chosen there). As a result when \( \{y < Y(\xi) \leq y + \Delta y\} \), the r.v \( X \) could be in any one of the three mutually exclusive intervals

\[ \{x_1 < X(\xi) \leq x_1 + \Delta x_1\}, \ \{x_2 + \Delta x_2 < X(\xi) \leq x_2\} \ \text{or} \ \{x_3 < X(\xi) \leq x_3 + \Delta x_3\}. \]

Hence the probability of the event in (5-26) is the sum of the probability of the above three events, i.e.,

\[ P\{y < Y(\xi) \leq y + \Delta y\} = P\{x_1 < X(\xi) \leq x_1 + \Delta x_1\} \]

\[ + P\{x_2 + \Delta x_2 < X(\xi) \leq x_2\} + P\{x_3 < X(\xi) \leq x_3 + \Delta x_3\}. \] (5-27)
For small $\Delta y, \Delta x_i$, making use of the approximation in (5-26), we get
\[
f_Y(y)\Delta y = f_X(x_1)\Delta x_1 + f_X(x_2)(-\Delta x_2) + f_X(x_3)\Delta x_3. \tag{5-28}
\]

In this case, $\Delta x_1 > 0$, $\Delta x_2 < 0$ and $\Delta x_3 > 0$, so that (5-28) can be rewritten as
\[
f_Y(y) = \sum_i f_X(x_i) \frac{\Delta x_i}{\Delta y} = \sum_i \frac{1}{|\Delta y/\Delta x_i|} f_X(x_i) \tag{5-29}
\]
and as $\Delta y \to 0$, (5-29) can be expressed as
\[
f_Y(y) = \sum_i \frac{1}{|dy/dx|_{x_i}} f_X(x_i) = \sum_i \frac{1}{|g'(x_i)|} f_X(x_i). \tag{5-30}
\]
The summation index $i$ in (5-30) depends on $y$, and for every $y$ the equation $y = g(x_i)$ must be solved to obtain the total number of solutions at every $y$, and the actual solutions $x_1, x_2, \ldots$ all in terms of $y$. 
For example, if $Y = X^2$, then for all $y > 0$, $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$ represent the two solutions for each $y$. Notice that the solutions $x_i$ are all in terms of $y$ so that the right side of (5-30) is only a function of $y$. Referring back to the example $Y = X^2$ (Example 5.2) here for each $y > 0$, there are two solutions given by $x_1 = -\sqrt{y}$ and $x_2 = +\sqrt{y}$. ($f_Y(y) = 0$ for $y < 0$).

Moreover
\[
\frac{dy}{dx} = 2x \quad \text{so that} \quad \left| \frac{dy}{dx} \right|_{x=x_i} = 2\sqrt{y}
\]

and using (5-30) we get
\[
f_Y(y) = \begin{cases} 
\frac{1}{2\sqrt{y}} \left( f_X(\sqrt{y}) + f_X(-\sqrt{y}) \right), & y > 0, \\
0, & \text{otherwise} 
\end{cases} \quad (5-31)
\]

which agrees with (5-14).
Example 5.5: \( Y = \frac{1}{X} \). Find \( f_Y(y) \).

Solution: Here for every \( y \), \( x_1 = 1/y \) is the only solution, and

\[
\frac{dy}{dx} = -\frac{1}{x^2} \quad \text{so that} \quad \left| \frac{dy}{dx} \right|_{x=x_1} = \frac{1}{1/y^2} = y^2,
\]

and substituting this into (5-30), we obtain

\[
f_Y(y) = \frac{1}{y^2} f_X \left( \frac{1}{y} \right). \tag{5-33}
\]

In particular, suppose \( X \) is a Cauchy r.v as in (3-39) with parameter \( \alpha \) so that

\[
f_X(x) = \frac{\alpha / \pi}{\alpha^2 + x^2}, \quad -\infty < x < +\infty. \tag{5-34}
\]

In that case from (5-33), \( Y = 1/X \) has the p.d.f

\[
f_Y(y) = \frac{1}{y^2} \frac{\alpha / \pi}{\alpha^2 + (1/y)^2} = \frac{(1/\alpha)/\pi}{(1/\alpha)^2 + y^2}, \quad -\infty < y < +\infty. \tag{5-35}
\]
But (5-35) represents the p.d.f of a Cauchy r.v with parameter $1/\alpha$. Thus if $X \sim C(\alpha)$, then $1/X \sim C(1/\alpha)$.

**Example 5.6:** Suppose $f_x(x) = 2x / \pi^2$, $0 < x < \pi$, and $Y = \sin X$. Determine $f_y(y)$.

Solution: Since $X$ has zero probability of falling outside the interval $(0, \pi)$, $y = \sin x$ has zero probability of falling outside the interval $(0,1)$. Clearly $f_y(y) = 0$ outside this interval. For any $0 < y < 1$, from Fig.5.6(b), the equation $y = \sin x$ has an infinite number of solutions $\cdots, x_1, x_2, x_3, \cdots$, where $x_1 = \sin^{-1} y$ is the principal solution. Moreover, using the symmetry we also get $x_2 = \pi - x_1$ etc. Further,

$$\frac{dy}{dx} = \cos x = \sqrt{1 - \sin^2 x} = \sqrt{1 - y^2}$$

so that

$$\left| \frac{dy}{dx} \right|_{x=x_i} = \sqrt{1 - y^2}.$$
Using this in (5-30), we obtain for $0 < y < 1$,

$$f_Y(y) = \sum_{i=\pm\infty, i\neq 0}^{+\infty} \frac{1}{\sqrt{1 - y^2}} f_X(x_i).$$  \hspace{1cm} (5-36)$$

But from Fig. 5.6(a), in this case $f_X(x_{-1}) = f_X(x_3) = f_X(x_4) = \cdots = 0$

(Except for $f_X(x_1)$ and $f_X(x_2)$ the rest are all zeros).
Thus (Fig. 5.7)

\[
f_Y(y) = \frac{1}{\sqrt{1 - y^2}} (f_x(x_1) + f_x(x_2)) = \frac{1}{\sqrt{1 - y^2}} \left( \frac{2x_1}{\pi^2} + \frac{2x_2}{\pi^2} \right)
\]

\[
= \frac{2(x_1 + \pi - x_1)}{\pi^2 \sqrt{1 - y^2}} = \begin{cases} 
\frac{2}{\pi \sqrt{1 - y^2}}, & 0 < y < 1, \\
0, & \text{otherwise.}
\end{cases}
\]

(5-37)

**Example 5.7:** Let \( Y = \tan X \) where \( X \sim U(-\pi/2, \pi/2) \).

Determine \( f_Y(y) \).

Solution: As \( x \) moves from \((-\pi/2, \pi/2)\), \( y \) moves from \((-\infty, +\infty)\).

From Fig.5.8(b), the function \( Y = \tan X \) is one-to-one for \(-\pi/2 < x < \pi/2\). For any \( y \), \( x_1 = \tan^{-1} y \) is the principal solution. Further

\[
\frac{dy}{dx} = \frac{d}{dx} \tan x = \sec^2 x = 1 + \tan^2 x = 1 + y^2
\]
so that using (5-30)

\[
f_Y(y) = \frac{1}{\left| \frac{dy}{dx} \right|_{x=x_1}} f_X(x_1) = \frac{1/\pi}{1 + y^2}, \quad -\infty < y < +\infty, \quad (5-38)
\]

which represents a Cauchy density function with parameter equal to unity (Fig. 5.9).
Functions of a discrete-type r.v

Suppose $X$ is a discrete-type r.v with

$$P(X = x_i) = p_i, \quad x = x_1, x_2, \ldots, x_i, \ldots$$

(5-39)

and $Y = g(X)$. Clearly $Y$ is also of discrete-type, and when $x = x_i$, $y_i = g(x_i)$, and for those $y_i$

$$P(Y = y_i) = P(X = x_i) = p_i, \quad y = y_1, y_2, \ldots, y_i, \ldots$$

(5-40)

**Example 5.8:** Suppose $X \sim P(\lambda)$, so that

$$P(X = k) = e^{-\lambda} \frac{\lambda^k}{k!}, \quad k = 0, 1, 2, \ldots$$

(5-41)

Define $Y = X^2 + 1$. Find the p.m.f of $Y$.

Solution: $X$ takes the values $0, 1, 2, \ldots, k, \ldots$ so that $Y$ only takes the value $1, 2, 5, \ldots, k^2 + 1, \ldots$ and
\[ P(Y = k^2 + 1) = P(X = k) \]

so that for \( j = k^2 + 1 \)

\[ P(Y = j) = P\left( X = \sqrt{j - 1} \right) = e^{-\lambda} \frac{\lambda^{\sqrt{j-1}}}{(\sqrt{j-1})!}, \quad j = 1, 2, 5, \ldots, k^2 + 1, \ldots \quad (5-42) \]