7. Two Random Variables

In many experiments, the observations are expressible not as a single quantity, but as a family of quantities. For example to record the height and weight of each person in a community or the number of people and the total income in a family, we need two numbers.

Let X and Y denote two random variables (r.v) based on a probability model (Ω, F, P). Then

$$P\left(x_1 < X(\xi) \le x_2\right) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)\,dx,$$

and

$$P\left(y_1 < Y(\xi) \le y_2\right) = F_Y(y_2) - F_Y(y_1) = \int_{y_1}^{y_2} f_Y(y)\,dy.$$

What about the probability that the pair of r.vs (X,Y) belongs to an arbitrary region D? In other words, how does one estimate, for example, $P\left[(x_1 < X(\xi) \le x_2) \cap (y_1 < Y(\xi) \le y_2)\right] = ?$ Towards this, we define the joint probability distribution function of X and Y to be

$$F_{XY}(x,y) = P\left[(X(\xi) \le x) \cap (Y(\xi) \le y)\right]$$
$$= P(X \le x, Y \le y) \ge 0, \qquad (7\text{-}1)$$

where x and y are arbitrary real numbers.

Properties

(i) $F_{XY}(-\infty, y) = F_{XY}(x, -\infty) = 0, \quad F_{XY}(+\infty, +\infty) = 1.$ (7-2)

since $\left(X(\xi) \le -\infty, Y(\xi) \le y\right) \subset \left(X(\xi) \le -\infty\right),$ we get

$F_{XY}(-\infty, y) \le P\big(X(\xi) \le -\infty\big) = 0.$ Similarly $\big(X(\xi) \le +\infty, Y(\xi) \le +\infty\big) = \Omega,$

we get $F_{XY}(\infty, \infty) = P(\Omega) = 1.$

(ii) $P\big(x_1 < X(\xi) \le x_2, Y(\xi) \le y\big) = F_{XY}(x_2, y) - F_{XY}(x_1, y).$ (7-3)

$P\big(X(\xi) \le x, y_1 < Y(\xi) \le y_2\big) = F_{XY}(x, y_2) - F_{XY}(x, y_1).$ (7-4)

To prove (7-3), we note that for $x_2 > x_1,$

$$\big(X(\xi) \le x_2, Y(\xi) \le y\big) = \big(X(\xi) \le x_1, Y(\xi) \le y\big) \cup \big(x_1 < X(\xi) \le x_2, Y(\xi) \le y\big)$$

and the mutually exclusive property of the events on the right side gives

$$P\big(X(\xi) \le x_2, Y(\xi) \le y\big) = P\big(X(\xi) \le x_1, Y(\xi) \le y\big) + P\big(x_1 < X(\xi) \le x_2, Y(\xi) \le y\big)$$

which proves (7-3). Similarly (7-4) follows.

(iii) $$P\big(x_1 < X(\xi) \le x_2,\, y_1 < Y(\xi) \le y_2\big) = F_{XY}(x_2, y_2) - F_{XY}(x_2, y_1) - F_{XY}(x_1, y_2) + F_{XY}(x_1, y_1). \qquad (7\text{-}5)$$

This is the probability that (X,Y) belongs to the rectangle R_0 in Fig. 7.1. To prove (7-5), we can make use of the following identity involving mutually exclusive events on the right side.

$$\big(x_1 < X(\xi) \le x_2, Y(\xi) \le y_2\big) = \big(x_1 < X(\xi) \le x_2, Y(\xi) \le y_1\big) \cup \big(x_1 < X(\xi) \le x_2, y_1 < Y(\xi) \le y_2\big).$$

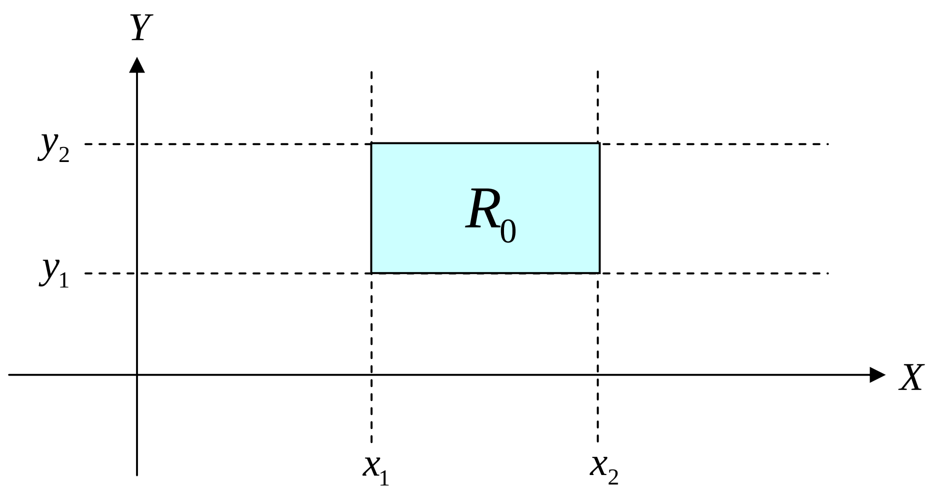

Fig. 7.1

This gives

$$P\left(x_1 < X(\xi) \le x_2, Y(\xi) \le y_2\right) = P\left(x_1 < X(\xi) \le x_2, Y(\xi) \le y_1\right) + P\left(x_1 < X(\xi) \le x_2, y_1 < Y(\xi) \le y_2\right)$$

and the desired result in (7-5) follows by making use of (7-3) with $y = y_2$ and y_1 respectively.

Joint probability density function (Joint p.d.f)

By definition, the joint p.d.f of X and Y is given by

$$f_{XY}(x,y) = \frac{\partial^2 F_{XY}(x,y)}{\partial x\, \partial y}. \qquad (7\text{-}6)$$

and hence we obtain the useful formula

$$F_{XY}(x,y) = \int_{-\infty}^{x} \int_{-\infty}^{y} f_{XY}(u,v)\, du dv\, . \qquad (7\text{-}7)$$

Using (7-2), we also get

$$\int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} f_{XY}(x,y)\, dx dy = 1. \qquad (7\text{-}8)$$

To find the probability that (X,Y) belongs to an arbitrary region D, we can make use of (7-5) and (7-7). From (7-5) and (7-7)

$$
\begin{aligned}
P\big(x < X(\xi) \le x+\Delta x,\, y < Y(\xi) \le y+\Delta y\big) &= F_{XY}(x+\Delta x, y+\Delta y) \\
&\quad - F_{XY}(x, y+\Delta y) - F_{XY}(x+\Delta x, y) + F_{XY}(x, y) \\
&= \int_x^{x+\Delta x}\int_y^{y+\Delta y} f_{XY}(u,v)\,du\,dv = f_{XY}(x,y)\Delta x \Delta y.
\end{aligned} \tag{7-9}
$$

Thus the probability that (X,Y) belongs to a differential rectangle $\Delta x\ \Delta y$ equals $f_{XY}(x,y)\cdot \Delta x \Delta y,$ and repeating this procedure over the union of no overlapping differential rectangles in D, we get the useful result

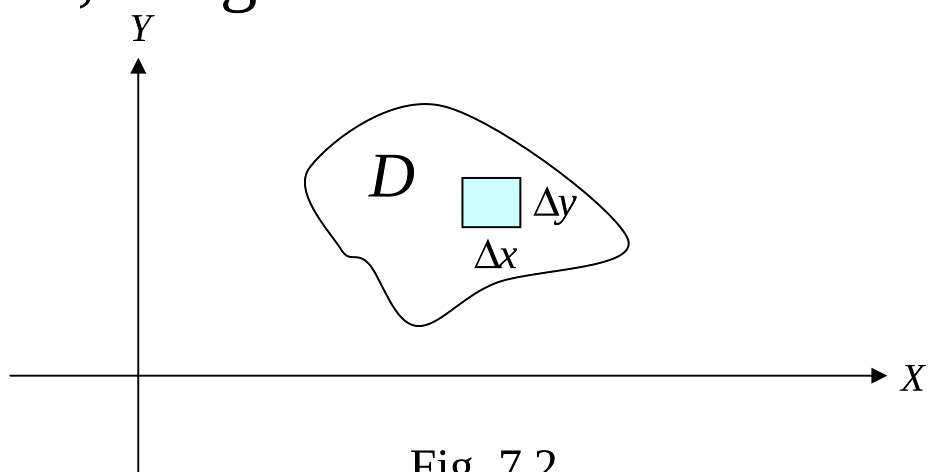

Fig. 7.2

$$P\big((X,Y)\in D\big)=\int\int_{(x,y)\in D} f_{XY}(x,y)\,dxdy\ . \qquad (7\text{-}10)$$

(iv) Marginal Statistics

In the context of several r.vs, the statistics of each individual ones are called marginal statistics. Thus $F_X(x)$ is the marginal probability distribution function of X, and $f_X(x)$ is the marginal p.d.f of X. It is interesting to note that all marginals can be obtained from the joint p.d.f. In fact

$$F_X(x)=F_{XY}(x,+\infty), \quad F_Y(y)=F_{XY}(+\infty,y). \qquad (7\text{-}11)$$

Also

$$f_X(x)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dy, \quad f_Y(y)=\int_{-\infty}^{+\infty} f_{XY}(x,y)\,dx. \qquad (7\text{-}12)$$

To prove (7-11), we can make use of the identity

$$(X\le x)=(X\le x)\cap(Y\le+\infty)$$

so that $F_X(x) = P(X \le x) = P(X \le x, Y \le \infty) = F_{XY}(x, +\infty).$

To prove (7-12), we can make use of (7-7) and (7-11), which gives

$$F_X(x) = F_{XY}(x, +\infty) = \int_{-\infty}^{x} \int_{-\infty}^{+\infty} f_{XY}(u, y)\, du dy \qquad (7\text{-}13)$$

and taking derivative with respect to *x* in (7-13), we get

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x, y) dy. \qquad (7\text{-}14)$$

At this point, it is useful to know the formula for differentiation under integrals. Let

$$H(x) = \int_{a(x)}^{b(x)} h(x, y) dy. \qquad (7\text{-}15)$$

Then its derivative with respect to *x* is given by

$$\frac{dH(x)}{dx} = \frac{db(x)}{dx} h(x, b) - \frac{da(x)}{dx} h(x, a) + \int_{a(x)}^{b(x)} \frac{\partial h(x, y)}{\partial x} dy. \qquad (7\text{-}16)$$

Obvious use of (7-16) in (7-13) gives (7-14).

If X and Y are discrete r.vs, then $p_{ij} \overset{\Delta}{=} P(X = x_i, Y = y_j)$ represents their joint p.d.f, and their respective marginal p.d.fs are given by

$$P(X = x_i) = \sum_j P(X = x_i, Y = y_j) = \sum_j p_{ij} \tag{7-17}$$

and

$$P(Y = y_j) = \sum_i P(X = x_i, Y = y_j) = \sum_i p_{ij} \tag{7-18}$$

Assuming that $P(X = x_i, Y = y_j)$ is written out in the form of a rectangular array, to obtain $P(X = x_i),$ from (7-17), one need to add up all entries in the i-th row.

It used to be a practice for insurance companies routinely to scribble out these sum values in the left and top margins, thus suggesting the name marginal densities! (Fig 7.3).

$$\begin{array}{c|cccccc}
 & & & \sum_i p_{ij} & & & \\
\hline
 & p_{11} & p_{12} & \cdots & p_{1j} & \cdots & p_{1n} \\
 & p_{21} & p_{22} & \cdots & p_{2j} & \cdots & p_{2n} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\sum_j p_{ij} & p_{i1} & p_{i2} & \cdots & p_{ij} & \cdots & p_{in} \\
 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
 & p_{m1} & p_{m2} & \cdots & p_{mj} & \cdots & p_{mn}
\end{array}$$

Fig. 7.3

From (7-11) and (7-12), the joint P.D.F and/or the joint p.d.f represent complete information about the r.vs, and their marginal p.d.fs can be evaluated from the joint p.d.f. However, given marginals, (most often) it will not be possible to compute the joint p.d.f. Consider the following example:

Example 7.1: Given

$$f_{XY}(x,y) = \begin{cases} \text{constant}, & 0 < x < y < 1, \\ 0, & \text{otherwise} \end{cases}. \qquad (7\text{-}19)$$

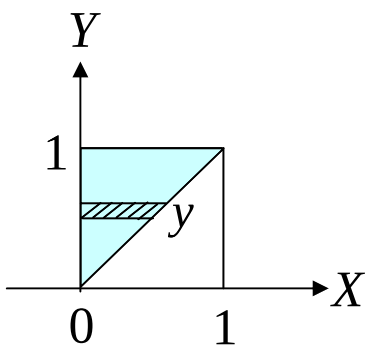

Fig. 7.4

Obtain the marginal p.d.fs $f_X(x)$ and $f_Y(y)$.

Solution: It is given that the joint p.d.f $f_{XY}(x,y)$ is a constant in the shaded region in Fig. 7.4. We can use (7-8) to determine that constant *c*. From (7-8)

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f_{XY}(x,y)dxdy = \int_{y=0}^{1}\left(\int_{x=0}^{y} c\cdot dx\right)dy = \int_{y=0}^{1} cydy = \frac{cy^2}{2}\Bigg|_0^1 = \frac{c}{2} = 1. \qquad (7\text{-}20)$$

Thus $c = 2$. Moreover from (7-14)

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dy = \int_{y=x}^{1} 2dy = 2(1-x), \quad 0 < x < 1, \qquad (7\text{-}21)$$

and similarly

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dx = \int_{x=0}^{y} 2dx = 2y, \quad 0 < y < 1. \qquad (7\text{-}22)$$

Clearly, in this case given $f_X(x)$ and $f_Y(y)$ as in (7-21)-(7-22), it will not be possible to obtain the original joint p.d.f in (7-19).

Example 7.2: X and Y are said to be jointly normal (Gaussian) distributed, if their joint p.d.f has the following form:

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} e^{\frac{-1}{2(1-\rho^2)}\left(\frac{(x-\mu_X)^2}{\sigma_X^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y} + \frac{(y-\mu_Y)^2}{\sigma_Y^2}\right)}, \qquad (7\text{-}23)$$

$$-\infty < x < +\infty, \; -\infty < y < +\infty, \; |\rho| < 1.$$

By direct integration, using (7-14) and completing the square in (7-23), it can be shown that

$$f_X(x) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dy = \frac{1}{\sqrt{2\pi\sigma_X^2}} e^{-(x-\mu_X)^2/2\sigma_X^2} \sim N(\mu_X, \sigma_X^2), \qquad (7\text{-}24)$$

and similarly

$$f_Y(y) = \int_{-\infty}^{+\infty} f_{XY}(x,y)dx = \frac{1}{\sqrt{2\pi\sigma_Y^2}} e^{-(y-\mu_Y)^2/2\sigma_Y^2} \sim N(\mu_Y, \sigma_Y^2), \qquad (7\text{-}25)$$

Following the above notation, we will denote (7-23) as $N(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho)$. Once again, knowing the marginals in (7-24) and (7-25) alone doesn't tell us everything about the joint p.d.f in (7-23).

As we show below, the only situation where the marginal p.d.fs can be used to recover the joint p.d.f is when the random variables are statistically independent.

Independence of r.vs

Definition: The random variables X and Y are said to be statistically independent if the events $\{X(\xi) \in A\}$ and $\{Y(\xi) \in B\}$ are independent events for any two Borel sets A and B in x and y axes respectively. Applying the above definition to the events $\{X(\xi) \le x\}$ and $\{Y(\xi) \le y\}$, we conclude that, if the r.vs X and Y are independent, then

$$P\big((X(\xi) \le x) \cap (Y(\xi) \le y)\big) = P(X(\xi) \le x)P(Y(\xi) \le y) \qquad (7\text{-}26)$$

i.e.,

$$F_{XY}(x, y) = F_X(x)F_Y(y) \qquad (7\text{-}27)$$

or equivalently, if X and Y are independent, then we must have

$$f_{XY}(x, y) = f_X(x)\, f_Y(y). \qquad (7\text{-}28)$$

If X and Y are discrete-type r.vs then their independence implies

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j) \quad \text{for all } \ i, j. \qquad (7\text{-}29)$$

Equations (7-26)-(7-29) give us the procedure to test for independence. Given $f_{XY}(x,y)$, obtain the marginal p.d.fs $f_X(x)$ and $f_Y(y)$ and examine whether (7-28) or (7-29) is valid. If so, the r.vs are independent, otherwise they are dependent. Returning back to Example 7.1, from (7-19)-(7-22), we observe by direct verification that $f_{XY}(x,y) \neq f_X(x) f_Y(y)$. Hence X and Y are dependent r.vs in that case. It is easy to see that such is the case in the case of Example 7.2 also, unless $\rho = 0$. In other words, two jointly Gaussian r.vs as in (7-23) are independent if and only if the fifth parameter $\rho = 0$.

Example 7.3: Given

$$f_{XY}(x,y) = \begin{cases} xy^2 e^{-y}, & 0 < y < \infty, \quad 0 < x < 1, \\ 0, & \text{otherwise.} \end{cases} \tag{7-30}$$

Determine whether *X* and *Y* are independent.

Solution:

$$f_X(x) = \int_0^{+\infty} f_{XY}(x,y)\,dy = x \int_0^{\infty} y^2 e^{-y}\,dy$$

$$= x\left(-2ye^{-y}\Big|_0^{\infty} + 2\int_0^{\infty} ye^{-y}\,dy \right) = 2x, \quad 0 < x < 1. \tag{7-31}$$

Similarly

$$f_Y(y) = \int_0^1 f_{XY}(x,y)\,dx = \frac{y^2}{2} e^{-y}, \quad 0 < y < \infty. \tag{7-32}$$

In this case

$$f_{XY}(x,y) = f_X(x)\, f_Y(y),$$

and hence *X* and *Y* are independent random variables.