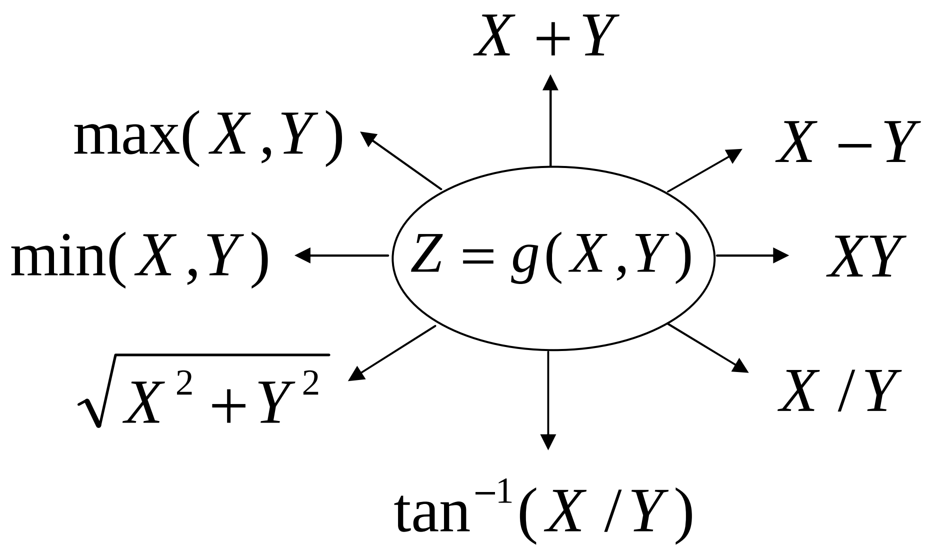
8. One Function of Two Random Variables

Given two random variables X and Y and a function $g(x,y)$, we form a new random variable Z as

$$Z = g(X, Y). \tag{8-1}$$

Given the joint p.d.f $f_{XY}(x, y)$, how does one obtain $f_Z(z)$, the p.d.f of Z ? Problems of this type are of interest from a practical standpoint. For example, a receiver output signal usually consists of the desired signal buried in noise, and the above formulation in that case reduces to $Z = X + Y$.

It is important to know the statistics of the incoming signal for proper receiver design. In this context, we shall analyze problems of the following type:



(8-2)

Referring back to (8-1), to start with

$$F_Z(z) = P\big(Z(\xi) \le z\big) = P\big(g(X,Y) \le z\big) = P\big[(X,Y) \in D_z\big]$$

$$= \int\int_{x,y \in D_z} f_{XY}(x,y)\,dx\,dy, \qquad (8\text{-}3)$$

where D_z in the *XY* plane represents the region such that $g(x,y) \le z$ is satisfied. Note that D_z need not be simply connected (Fig. 8.1). From (8-3), to determine $F_Z(z)$ it is enough to find the region D_z for every *z*, and then evaluate the integral there.

We shall illustrate this method through various examples.

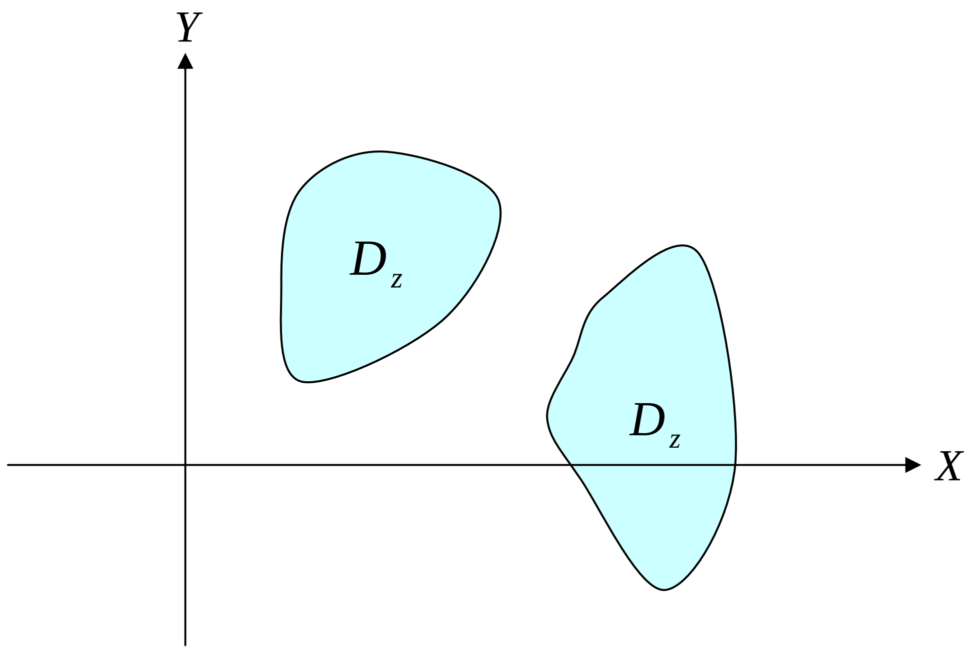


Fig. 8.1

Example 8.1: $Z = X + Y$. Find $f_Z(z)$.

Solution:

$$F_Z(z) = P\left(X + Y \le z\right) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{z-y} f_{XY}(x, y)\,dx\,dy, \tag{8-4}$$

since the region D_z of the *xy* plane where $x + y \le z$ is the shaded area in Fig. 8.2 to the left of the line $x + y = z$. Integrating over the horizontal strip along the *x*-axis first (inner integral) followed by sliding that strip along the *y*-axis from $-\infty$ to $+\infty$ (outer integral) we cover the entire shaded area.

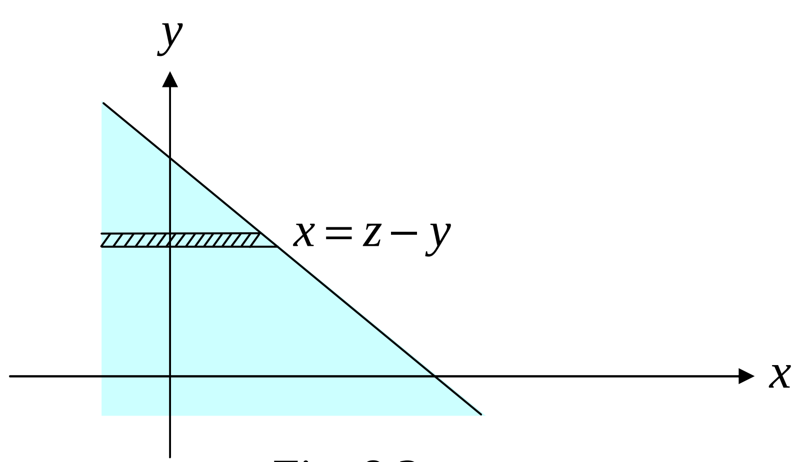


Fig. 8.2

We can find $f_Z(z)$ by differentiating $F_Z(z)$ directly. In this context, it is useful to recall the differentiation rule in (7-15) - (7-16) due to Leibnitz. Suppose

$$H(z) = \int_{a(z)}^{b(z)} h(x, z)\,dx. \tag{8-5}$$

Then

$$\frac{dH(z)}{dz} = \frac{db(z)}{dz} h\big(b(z), z\big) - \frac{da(z)}{dz} h\big(a(z), z\big) + \int_{a(z)}^{b(z)} \frac{\partial h(x, z)}{\partial z}\,dx. \tag{8-6}$$

Using (8-6) in (8-4) we get

$$f_Z(z) = \int_{-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{-\infty}^{z-y} f_{XY}(x, y)\,dx \right) dy = \int_{-\infty}^{+\infty} \left(f_{XY}(z-y, y) - 0 + \int_{-\infty}^{z-y} \frac{\partial f_{XY}(x, y)}{\partial z} \right) dy$$

$$= \int_{-\infty}^{+\infty} f_{XY}(z-y, y)\,dy. \tag{8-7}$$

Alternatively, the integration in (8-4) can be carried out first along the *y*-axis followed by the *x*-axis as in Fig. 8.3.

In that case

$$F_Z(z) = \int_{x=-\infty}^{+\infty} \int_{y=-\infty}^{z-x} f_{XY}(x,y)\,dx\,dy, \qquad \text{(8-8)}$$

and differentiation of (8-8) gives

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{x=-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{y=-\infty}^{z-x} f_{XY}(x,y)\,dy \right) dx$$

$$= \int_{x=-\infty}^{+\infty} f_{XY}(x, z-x)\,dx. \qquad \text{(8-9)}$$

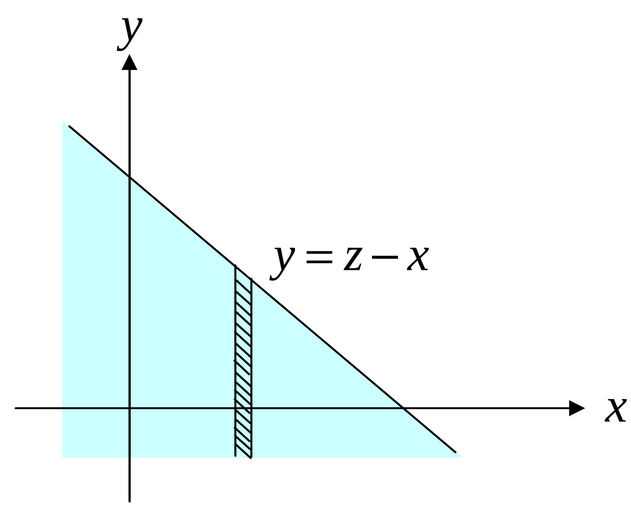


Fig. 8.3

If X and Y are independent, then

$$f_{XY}(x,y) = f_X(x)\, f_Y(y) \qquad \text{(8-10)}$$

and inserting (8-10) into (8-8) and (8-9), we get

$$f_Z(z) = \int_{y=-\infty}^{+\infty} f_X(z-y)\, f_Y(y)\,dy = \int_{x=-\infty}^{+\infty} f_X(x)\, f_Y(z-x)\,dx. \qquad \text{(8-11)}$$

The above integral is the standard convolution of the functions $f_X(z)$ and $f_Y(z)$ expressed two different ways. We thus reach the following conclusion: If two r.vs are independent, then the density of their sum equals the convolution of their density functions.

As a special case, suppose that $f_X(x) = 0$ for $x < 0$ and $f_Y(y) = 0$ for $y < 0,$ then we can make use of Fig. 8.4 to determine the new limits for $D_z.$

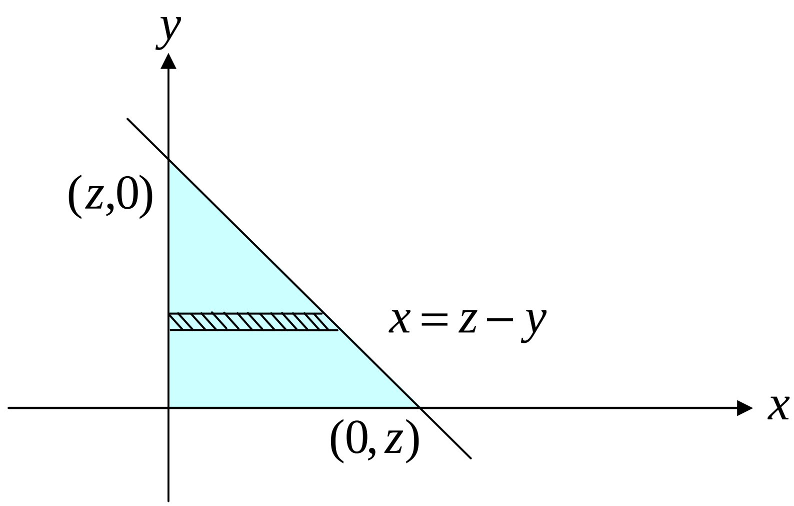


Fig. 8.4

In that case

$$F_Z(z) = \int_{y=0}^{z} \int_{x=0}^{z-y} f_{XY}(x,y)\,dx\,dy$$

or

$$f_Z(z) = \int_{y=0}^{z} \left(\frac{\partial}{\partial z} \int_{x=0}^{z-y} f_{XY}(x,y)\,dx \right) dy = \begin{cases} \int_0^z f_{XY}(z-y,y)\,dy, & z>0, \\ 0, & z \le 0. \end{cases} \qquad \text{(8-12)}$$

On the other hand, by considering vertical strips first in Fig. 8.4, we get

$$F_Z(z) = \int_{x=0}^{z} \int_{y=0}^{z-x} f_{XY}(x,y)\,dy\,dx$$

or

$$f_Z(z) = \int_{x=0}^{z} f_{XY}(x, z-x)\,dx = \begin{cases} \int_{y=0}^{z} f_X(x) f_Y(z-x)\,dx, & z>0, \\ 0, & z \le 0, \end{cases} \qquad \text{(8-13)}$$

if X and Y are independent random variables.

Example 8.2: Suppose *X* and *Y* are independent exponential r.vs with common parameter λ, and let *Z* = *X* + *Y*. Determine $f_Z(z)$.

Solution: We have $f_X(x) = \lambda e^{-\lambda x} U(x), \quad f_Y(y) = \lambda e^{-\lambda y} U(y),$ (8-14)

and we can make use of (13) to obtain the p.d.f of *Z* = *X* + *Y*.

$$f_Z(z) = \int_0^z \lambda^2 e^{-\lambda x} e^{-\lambda(z-x)} dx = \lambda^2 e^{-\lambda z} \int_0^z dx = z\lambda^2 e^{-\lambda z} U(z). \qquad (8\text{-}15)$$

As the next example shows, care should be taken in using the convolution formula for r.vs with finite range.

Example 8.3: *X* and *Y* are independent uniform r.vs in the common interval (0,1). Determine $f_Z(z)$, where *Z* = *X* + *Y*.

Solution: Clearly, $Z = X + Y \Rightarrow 0 < z < 2$ here, and as Fig. 8.5 shows there are two cases of *z* for which the shaded areas are quite different in shape and they should be considered separately.

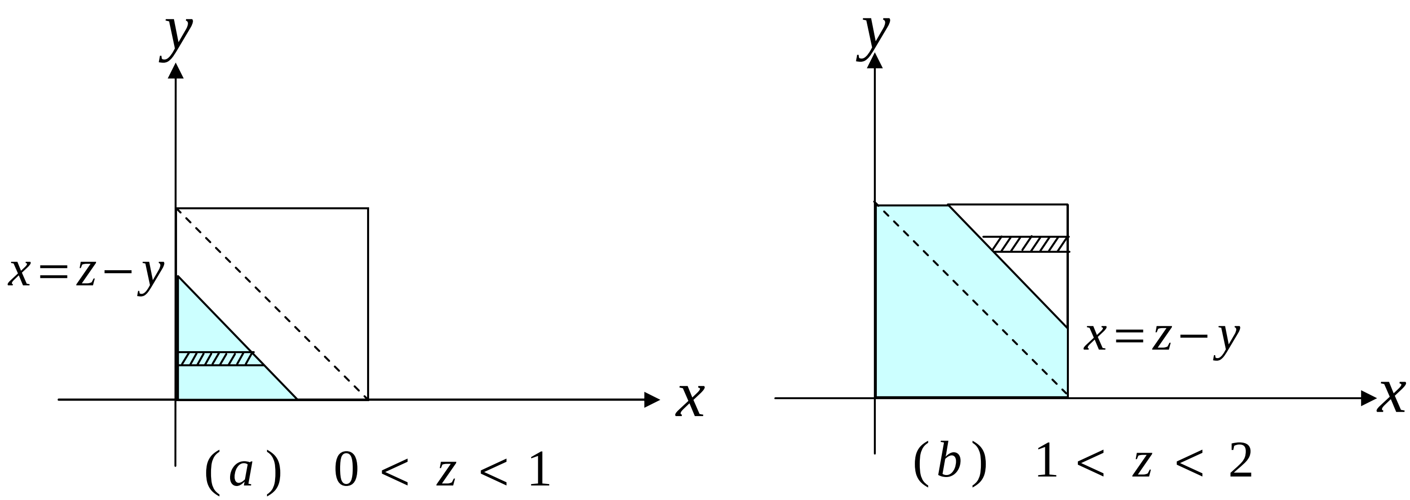


Fig. 8.5

For $0 \le z < 1,$

$$F_Z(z) = \int_{y=0}^{z} \int_{x=0}^{z-y} 1 \; dxdy = \int_{y=0}^{z} (z-y)dy = \frac{z^2}{2}, \quad 0 \le z < 1. \qquad \text{(8-16)}$$

For $1 \le z < 2,$ notice that it is easy to deal with the unshaded region. In that case

$$F_Z(z) = 1 - P(Z > z) = 1 - \int_{y=z-1}^{1} \int_{x=z-y}^{1} 1 \; dxdy$$

$$= 1 - \int_{y=z-1}^{1} (1 - z + y)dy = 1 - \frac{(2-z)^2}{2}, \quad 1 \le z < 2. \qquad \text{(8-17)}$$

Using (8-16) - (8-17), we obtain

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \begin{cases} z & 0 \le z < 1, \\ 2 - z, & 1 \le z < 2. \end{cases} \tag{8-18}$$

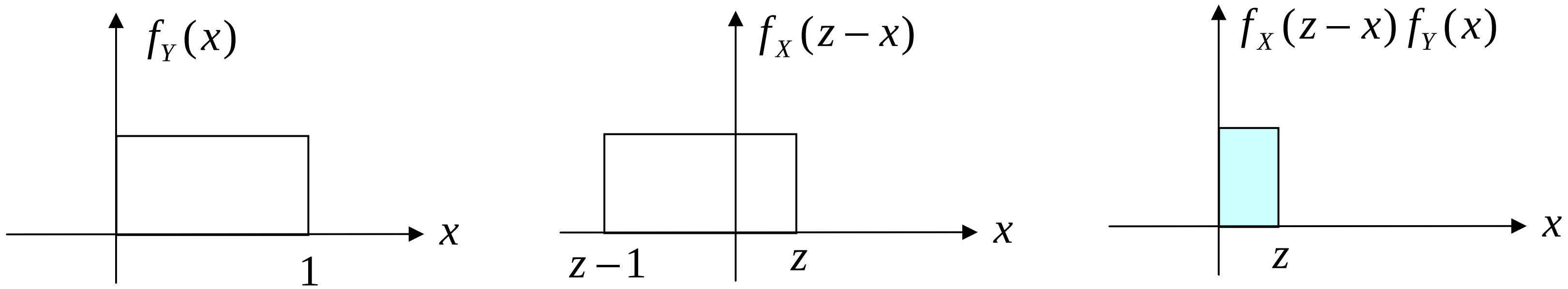
By direct convolution of $f_X(x)$ and $f_Y(y)$, we obtain the same result as above. In fact, for $0 \le z < 1$ (Fig. 8.6(a))

$$f_Z(z) = \int f_X(z-x) f_Y(x) dx = \int_0^z 1 \, dx = z. \tag{8-19}$$

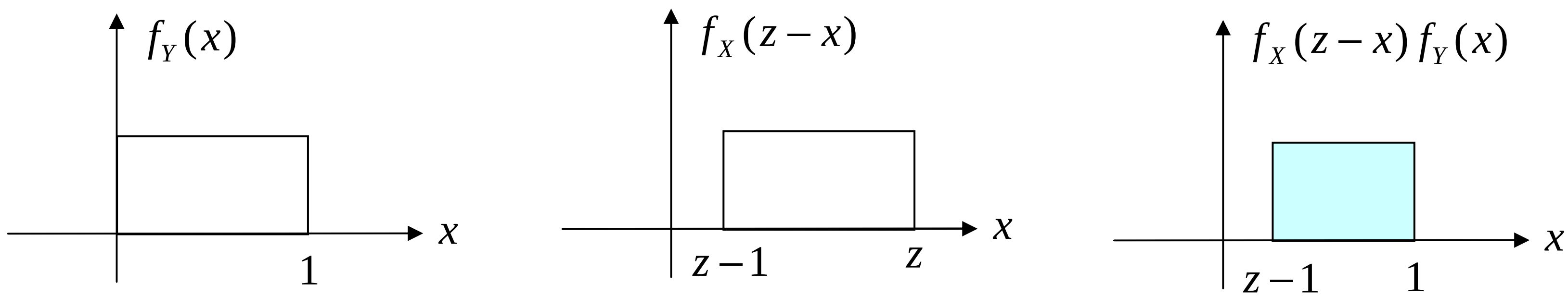
and for $1 \le z < 2$ (Fig. 8.6(b))

$$f_Z(z) = \int_{z-1}^1 1 \, dx = 2 - z. \tag{8-20}$$

Fig 8.6 (c) shows $f_Z(z)$ which agrees with the convolution of two rectangular waveforms as well.



$(a)\quad 0 \le z < 1$



$(b)\quad 1 \le z < 2$

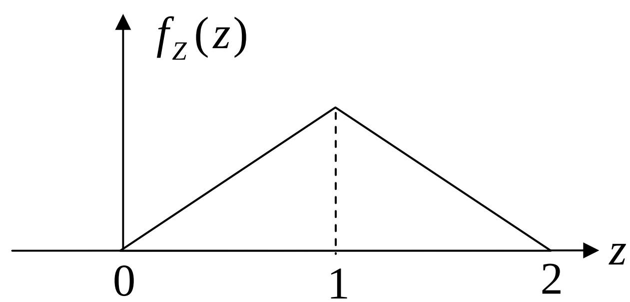


Fig. 8.6 (c)

Example 8.3: Let $Z = X - Y$. Determine its p.d.f $f_Z(z)$.

Solution: From (8-3) and Fig. 8.7

$$F_Z(z) = P(X - Y \le z) = \int_{y=-\infty}^{+\infty} \int_{x=-\infty}^{z+y} f_{XY}(x, y)\,dx\,dy$$

and hence

$$f_Z(z) = \frac{dF_Z(z)}{dz} = \int_{y=-\infty}^{+\infty} \left(\frac{\partial}{\partial z} \int_{x=-\infty}^{z+y} f_{XY}(x, y)\,dx \right) dy = \int_{-\infty}^{+\infty} f_{XY}(y+z, y)\,dy. \quad (8\text{-}21)$$

If X and Y are independent, then the above formula reduces to

$$f_Z(z) = \int_{-\infty}^{+\infty} f_X(z+y) f_Y(y)\,dy = f_X(-z) \otimes f_Y(y), \qquad (8\text{-}22)$$

which represents the convolution of $f_X(-z)$ with $f_Y(z)$.

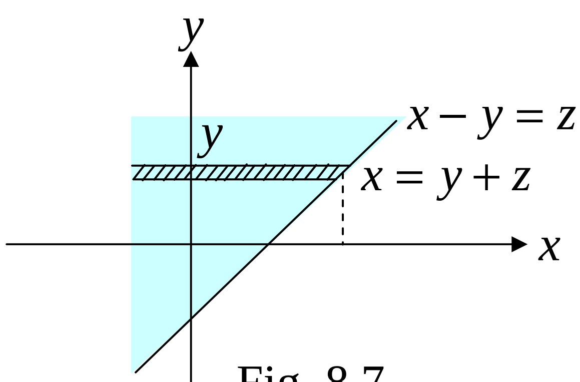


Fig. 8.7

As a special case, suppose

$$f_X(x)=0,\ \ x<0, \quad \text{and} \quad f_Y(y)=0,\ \ y<0.$$

In this case, Z can be negative as well as positive, and that gives rise to two situations that should be analyzed separately, since the region of integration for $z \geq 0$ and $z<0$ are quite different. For $z \geq 0$, from Fig. 8.8 (a)

$$F_Z(z)=\int_{y=0}^{+\infty}\int_{x=0}^{z+y} f_{XY}(x,y)dxdy$$

and for $z<0$, from Fig 8.8 (b)

$$F_Z(z)=\int_{y=-z}^{+\infty}\int_{x=0}^{z+y} f_{XY}(x,y)dxdy$$

After differentiation, this gives

$$f_Z(z)=\begin{cases}\int_0^{+\infty} f_{XY}(z+y,y)dy, & z \geq 0,\\ \int_{-z}^{+\infty} f_{XY}(z+y,y)dy, & z<0.\end{cases} \qquad (8\text{-}23)$$

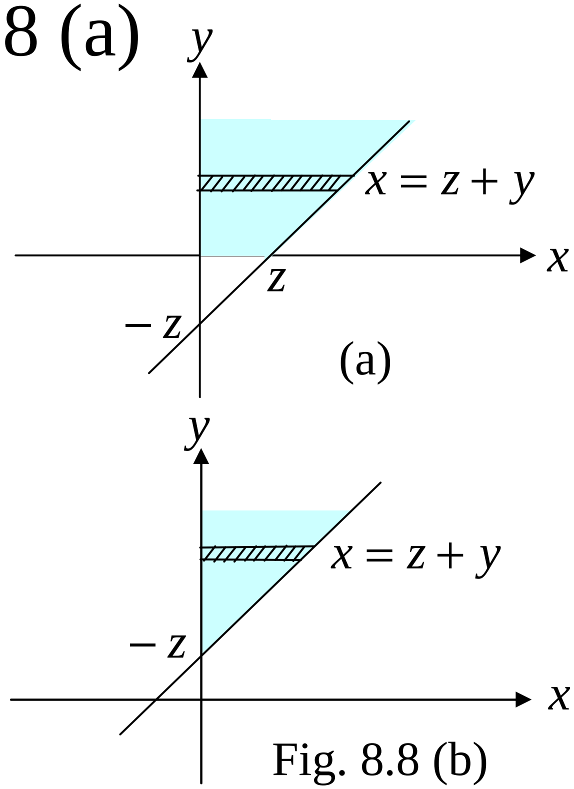


Fig. 8.8 (b)

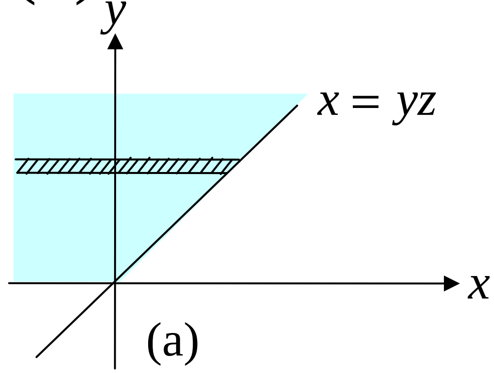
Example 8.4: Given $Z = X / Y$, obtain its density function.
Solution: We have $F_Z(z) = P(X / Y \le z)$. (8-24)
The inequality $X / Y \le z$ can be rewritten as $X \le Yz$ if $Y > 0$, and $X \ge Yz$ if $Y < 0$. Hence the event $(X / Y \le z)$ in (8-24) need to be conditioned by the event $A = (Y > 0)$ and its compliment $\overline{A}$. Since $A \cup \overline{A} = \Omega$, by the partition theorem, we have

$$\{X / Y \le z\} = \{(X / Y \le z) \cap (A \cup \overline{A})\} = \{(X / Y \le z) \cap A\} \cup \{(X / Y \le z) \cap \overline{A}\}$$

and hence by the mutually exclusive property of the later two events

$$\begin{aligned} P(X / Y \le z) &= P(X / Y \le z, Y > 0) + P(X / Y \le z, Y < 0) \\ &= P(X \le Yz, Y > 0) + P(X \ge Yz, Y < 0). \end{aligned} \quad (8\text{-}25)$$

Fig. 8.9(a) shows the area corresponding to the first term, and Fig. 8.9(b) shows that corresponding to the second term in (8-25).



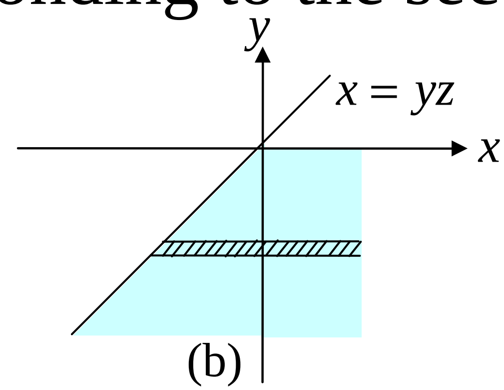


Fig. 8.9

Integrating over these two regions, we get

$$F_Z(z) = \int_{y=0}^{+\infty}\int_{x=-\infty}^{yz} f_{XY}(x,y)dxdy + \int_{y=-\infty}^{0}\int_{x=yz}^{\infty} f_{XY}(x,y)dxdy. \qquad (8\text{-}26)$$

Differentiation with respect to z gives

$$f_Z(z) = \int_0^{+\infty} y f_{XY}(yz,y)dy + \int_{-\infty}^{0}(-y) f_{XY}(yz,y)dy$$

$$= \int_{-\infty}^{+\infty} |\,y\,|\, f_{XY}(yz,y)dy, \qquad -\infty < z < +\infty. \qquad (8\text{-}27)$$

Note that if X and Y are nonnegative random variables, then the area of integration reduces to that shown in Fig. 8.10.

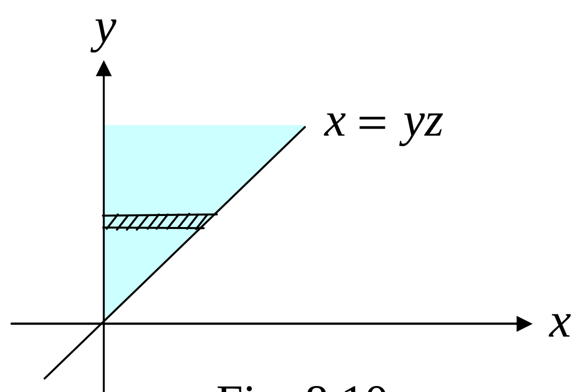


Fig. 8.10

This gives

$$F_Z(z) = \int_{y=0}^{\infty} \int_{x=0}^{yz} f_{XY}(x,y)dxdy$$

or

$$f_Z(z) = \begin{cases} \int_0^{+\infty} y\ f_{XY}(yz,y)dy, & z > 0, \\ 0, & \text{otherwise.} \end{cases} \tag{8-28}$$

Example 8.5: *X* and *Y* are jointly normal random variables with zero mean so that

$$f_{XY}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} e^{-\frac{1}{2(1-r^2)}\left(\frac{x^2}{\sigma_1^2} - \frac{2rxy}{\sigma_1\sigma_2} + \frac{y^2}{\sigma_2^2}\right)}. \tag{8-29}$$

Show that the ratio $Z = X / Y$ has a Cauchy density function centered at $r\sigma_1/\sigma_2$.

Solution: Inserting (8-29) into (8-27) and using the fact that $f_{XY}(-x,-y) = f_{XY}(x,y)$, we obtain

$$f_Z(z) = \frac{2}{2\pi\sigma_1\sigma_2\sqrt{1-r^2}} \int_0^{\infty} y e^{-y^2/2\sigma_0^2} dy = \frac{\sigma_0^2(z)}{\pi\sigma_1\sigma_2\sqrt{1-r^2}},$$

where

$$\sigma_0^2(z) = \frac{1-r^2}{\dfrac{z^2}{\sigma_1^2} - \dfrac{2rz}{\sigma_1\sigma_2} + \dfrac{1}{\sigma_2^2}}.$$

Thus

$$f_Z(z) = \frac{\sigma_1\sigma_2\sqrt{1-r^2}\,/\pi}{\sigma_2^2(z - r\sigma_1/\sigma_2)^2 + \sigma_1^2(1-r^2)}, \qquad (8\text{-}30)$$

which represents a Cauchy r.v centered at $r\sigma_1/\sigma_2.$ Integrating (8-30) from $-\infty$ to z, we obtain the corresponding distribution function to be

$$F_Z(z) = \frac{1}{2} + \frac{1}{\pi}\arctan\frac{\sigma_2 z - r\sigma_1}{\sigma_1\sqrt{1-r^2}}. \qquad (8\text{-}31)$$

Example 8.6: $Z = X^2 + Y^2.$ Obtain $f_Z(z).$

Solution: We have

$$F_Z(z) = P\left(X^2 + Y^2 \le z\right) = \int\int_{X^2+Y^2\le z} f_{XY}(x,y)\,dxdy. \qquad (8\text{-}32)$$

But, $X^2+Y^2 \le z$ represents the area of a circle with radius $\sqrt{z}$, and hence from Fig. 8.11,

$$F_Z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \int_{x=-\sqrt{z-y^2}}^{\sqrt{z-y^2}} f_{XY}(x,y)\,dx\,dy. \tag{8-33}$$

This gives after repeated differentiation

$$f_Z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z-y^2}} \left(f_{XY}(\sqrt{z-y^2}, y) + f_{XY}(-\sqrt{z-y^2}, y) \right) dy. \tag{8-34}$$

As an illustration, consider the next example.

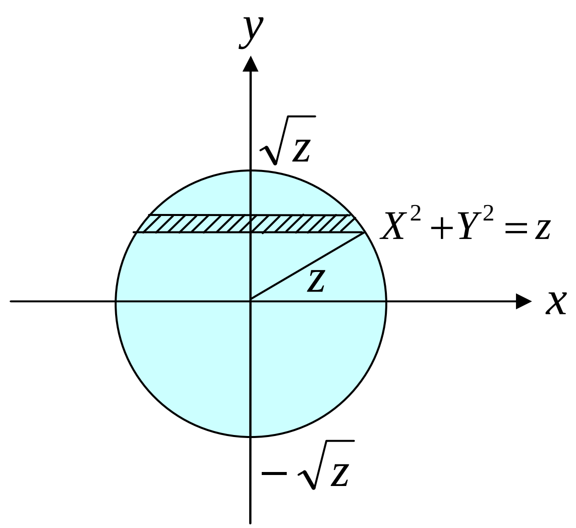


Fig. 8.11

Example 8.7 : X and Y are independent normal r.vs with zero Mean and common variance σ^2. Determine $f_Z(z)$ for $Z = X^2 + Y^2$.
Solution: Direct substitution of (8-29) with $r = 0, \quad \sigma_1 = \sigma_2 = \sigma$ Into (8-34) gives

$$f_Z(z) = \int_{y=-\sqrt{z}}^{\sqrt{z}} \frac{1}{2\sqrt{z - y^2}} \left(2 \cdot \frac{1}{2\pi\sigma^2} e^{-(z-y^2+y^2)/2\sigma^2} \right) dy = \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\sqrt{z}} \frac{1}{\sqrt{z - y^2}} \, dy$$

$$= \frac{e^{-z/2\sigma^2}}{\pi\sigma^2} \int_0^{\pi/2} \frac{\sqrt{z}\cos\theta}{\sqrt{z}\cos\theta} \, d\theta = \frac{1}{2\sigma^2} e^{-z/2\sigma^2} U(z), \tag{8-35}$$

where we have used the substitution $y = \sqrt{z}\sin\theta$. From (8-35) we have the following result: If X and Y are independent zero mean Gaussian r.vs with common variance σ^2, then $X^2 + Y^2$ is an exponential r.vs with parameter $2\sigma^2$.

Example 8.8 : Let $Z = \sqrt{X^2 + Y^2}$. Find $f_Z(z)$.
Solution: From Fig. 8.11, the present case corresponds to a circle with radius z^2. Thus

$$F_Z(z) = \int_{y=-z}^{z} \int_{x=-\sqrt{z^2-y^2}}^{\sqrt{z^2-y^2}} f_{XY}(x, y)\,dxdy\ .$$

And by repeated differentiation, we obtain

$$f_Z(z) = \int_{-z}^{z} \frac{z}{\sqrt{z^2 - y^2}} \left(f_{XY}(\sqrt{z^2 - y^2}, y) + f_{XY}(-\sqrt{z^2 - y^2}, y) \right) dy\ . \qquad (8\text{-}36)$$

Now suppose *X* and *Y* are independent Gaussian as in Example 8.7. In that case, (8-36) simplifies to

$$f_Z(z) = 2\int_0^z \frac{z}{\sqrt{z^2 - y^2}} \frac{1}{2\pi\sigma^2} e^{(z^2-y^2+y^2)/2\sigma^2} dy = \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_0^z \frac{1}{\sqrt{z^2 - y^2}}\, dy$$

$$= \frac{2z}{\pi\sigma^2} e^{-z^2/2\sigma^2} \int_0^{\pi/2} \frac{z\cos\theta}{z\cos\theta}\, d\theta = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2} U(z), \qquad (8\text{-}37)$$

which represents a Rayleigh distribution. Thus, if $W = X + iY$, where *X* and *Y* are real, independent normal r.vs with zero mean and equal variance, then the r.v $|W| = \sqrt{X^2 + Y^2}$ has a Rayleigh density. *W* is said to be a complex Gaussian r.v with zero mean, whose real and imaginary parts are independent r.vs. From (8-37), we have seen that its magnitude has Rayleigh distribution.

What about its phase

$$\theta = \tan^{-1}\left(\frac{X}{Y}\right)? \qquad (8\text{-}38)$$

Clearly, the principal value of θ lies in the interval $(-\pi/2, \pi/2)$. If we let $U = \tan\theta = X/Y$, then from example 8.5, U has a Cauchy distribution with (see (8-30) with $\sigma_1 = \sigma_2,\ r = 0$)

$$f_U(u) = \frac{1/\pi}{u^2+1}, \quad -\infty < u < \infty. \qquad (8\text{-}39)$$

As a result

$$f_\theta(\theta) = \frac{1}{|d\theta/du|} f_U(\tan\theta) = \frac{1}{(1/\sec^2\theta)} \frac{1/\pi}{\tan^2\theta+1} = \begin{cases} 1/\pi, & -\pi/2 < \theta < \pi/2, \\ 0, & \text{otherwise}. \end{cases} \qquad (8\text{-}40)$$

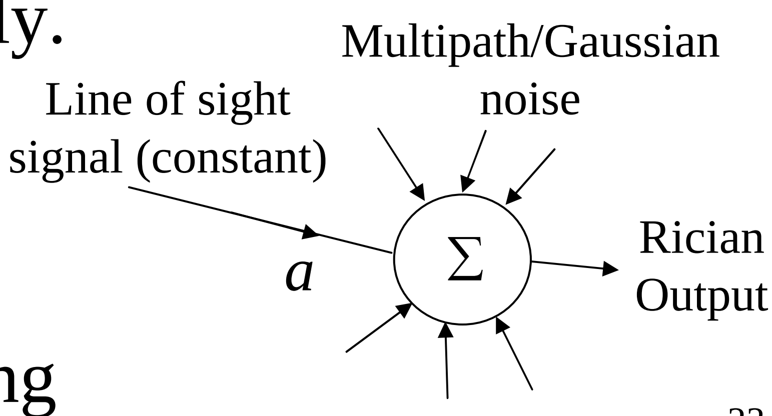
To summarize, the magnitude and phase of a zero mean complex Gaussian r.v has Rayleigh and uniform distributions respectively. Interestingly, as we will show later, these two derived r.vs are also independent of each other!

Let us reconsider example 8.8 where *X* and *Y* have nonzero means μ_X and μ_Y respectively. Then $Z = \sqrt{X^2 + Y^2}$ is said to be a Rician r.v. Such a scene arises in fading multipath situation where there is a dominant constant component (mean) in addition to a zero mean Gaussian r.v. The constant component may be the line of sight signal and the zero mean Gaussian r.v part could be due to random multipath components adding up incoherently (see diagram below). The envelope of such a signal is said to have a Rician p.d.f.

Example 8.9: Redo example 8.8, where *X* and *Y* have nonzero means μ_X and μ_Y respectively.

Solution: Since

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-[(x-\mu_X)^2 + (y-\mu_Y)^2]/2\sigma^2},$$



substituting this into (8-36) and letting

$$x = z\cos\theta,\ y = z\sin\theta,\quad \mu = \sqrt{\mu_X^2 + \mu_Y^2},\quad \mu_X = \mu\cos\phi,\quad \mu_Y = \mu\sin\phi,$$

we get the Rician probability density function to be

$$\begin{aligned} f_Z(z) &= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2}\int_{-\pi/2}^{\pi/2}\left(e^{z\mu\cos(\theta-\phi)/\sigma^2} + e^{-z\mu\cos(\theta+\phi)/\sigma^2}\right)d\theta \\ &= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2}\left(\int_{-\pi/2}^{\pi/2} e^{z\mu\cos(\theta-\phi)/\sigma^2}\,d\theta + \int_{\pi/2}^{3\pi/2} e^{z\mu\cos(\theta-\phi)/\sigma^2}\,d\theta\right) \\ &= \frac{ze^{-(z^2+\mu^2)/2\sigma^2}}{2\pi\sigma^2} I_0\left(\frac{z\mu}{\sigma^2}\right), \end{aligned} \qquad (8\text{-}41)$$

where

$$I_0(\eta) \triangleq \frac{1}{2\pi}\int_0^{2\pi} e^{\eta\cos(\theta-\phi)}d\theta = \frac{1}{\pi}\int_0^{\pi} e^{\eta\cos\theta}d\theta \qquad (8\text{-}42)$$

is the modified Bessel function of the first kind and zero$^{\text{th}}$ order.

Example 8.10: $Z = \max(X, Y),\quad W = \min(X, Y).$ Determine $f_Z(z)$.
Solution: The functions *max* and *min* are nonlinear

operators and represent special cases of the more general order statistics. In general, given any n-tuple $X_1, X_2, \cdots, X_n$, we can arrange them in an increasing order of magnitude such that

$$X_{(1)} \le X_{(2)} \le \cdots \le X_{(n)}, \tag{8-43}$$

where $X_{(1)} = \min(X_1, X_2, \cdots, X_n)$, and $X_{(2)}$ is the second smallest value among $X_1, X_2, \cdots, X_n$, and finally $X_{(n)} = \max(X_1, X_2, \cdots, X_n)$. If $X_1, X_2, \cdots, X_n$ represent r.vs, the function $X_{(k)}$ that takes on the value $x_{(k)}$ in each possible sequence $(x_1, x_2, \cdots, x_n)$ is known as the k-th order statistic. $(X_{(1)}, X_{(2)}, \cdots, X_{(n)})$ represent the set of order statistics among n random variables. In this context

$$R = X_{(n)} - X_{(1)} \tag{8-44}$$

represents the range, and when n = 2, we have the *max* and *min* statistics.

Returning back to that problem, since

$$Z = \max(X, Y) = \begin{cases} X, & X > Y, \\ Y, & X \le Y, \end{cases} \qquad \text{(8-45)}$$

we have (see also (8-25))

$$\begin{aligned} F_Z(z) &= P\big(\max(X, Y) \le z\big) = P\big[(X \le z, X > Y) \cup (Y \le z, X \le Y)\big] \\ &= P(X \le z, X > Y) + P(Y \le z, X \le Y), \end{aligned}$$

since $(X > Y)$ and $(X \le Y)$ are mutually exclusive sets that form a partition. Figs 8.12 (a)-(b) show the regions satisfying the corresponding inequalities in each term above.

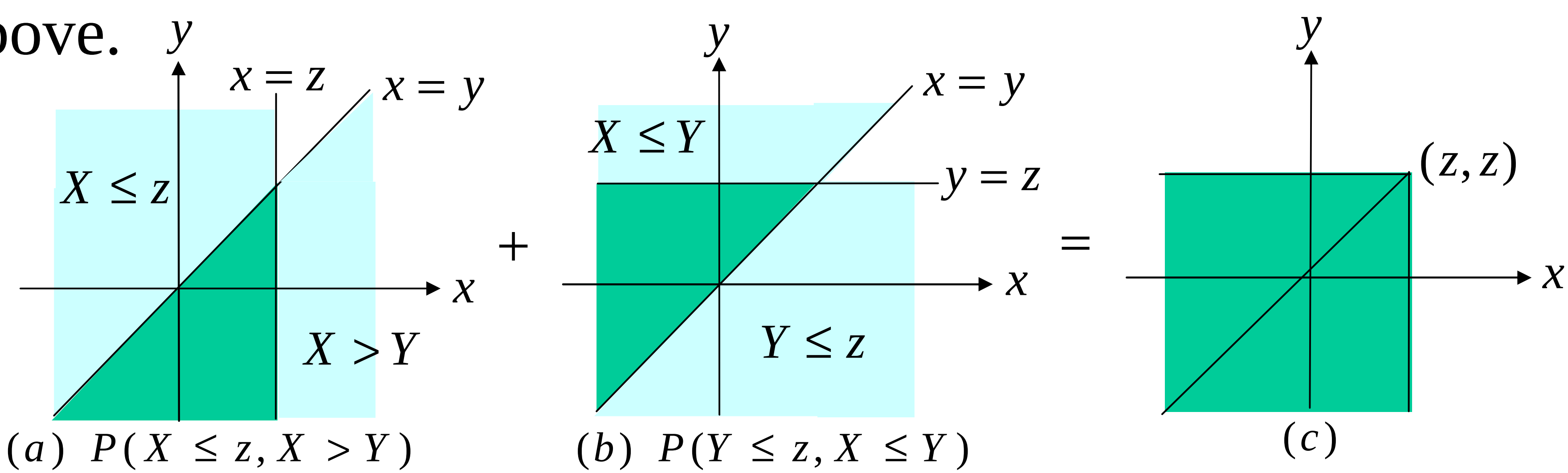


Fig. 8.12

Fig. 8.12 (c) represents the total region, and from there

$$F_Z(z) = P\left(X \le z, Y \le z\right) = F_{XY}(z, z). \tag{8-46}$$

If X and Y are independent, then

$$F_Z(z) = F_X(x) F_Y(y)$$

and hence

$$f_Z(z) = F_X(z) f_Y(z) + f_X(z) F_Y(z). \tag{8-47}$$

Similarly

$$W = \min(X, Y) = \begin{cases} Y, & X > Y, \\ X, & X \le Y. \end{cases} \tag{8-48}$$

Thus

$$F_W(w) = P\left(\min(X, Y) \le w\right) = P\left[\left(Y \le w, X > Y\right) \cup \left(X \le w, X \le Y\right)\right].$$

Once again, the shaded areas in Fig. 8.13 (a)-(b) show the regions satisfying the above inequalities and Fig 8.13 (c) shows the overall region.

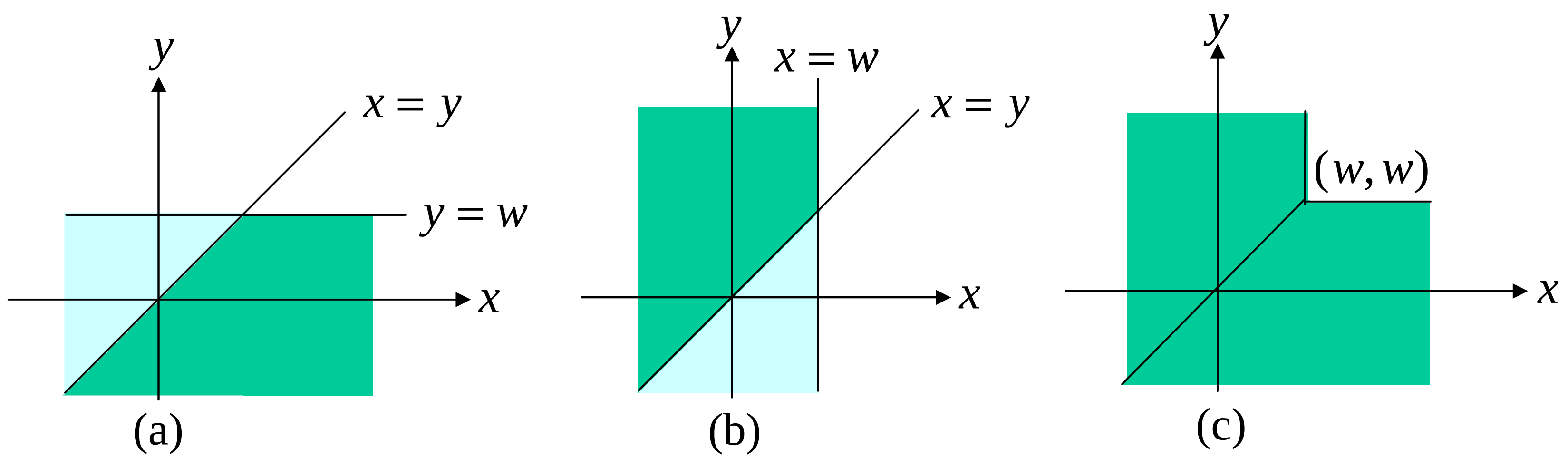


Fig. 8.13

From Fig. 8.13 (c),

$$F_W(w) = 1 - P(W > w) = 1 - P(X > w, Y > w)$$

$$= F_X(w) + F_Y(w) - F_{XY}(w, w), \qquad (8\text{-}49)$$

where we have made use of (7-5) and (7-12) with $x_2 = y_2 = +\infty$, and $x_1 = y_1 = w$.

Example 8.11: Let X and Y be independent exponential r.vs with common parameter λ. Define $W = \min(X, Y)$. Find $f_W(w)$?
Solution: From (8-49)

$$F_W(w) = F_X(w) + F_Y(w) - F_X(w)F_Y(w)$$

and hence

$$f_W(w) = f_X(w) + f_Y(w) - f_X(w)F_Y(w) - F_X(w)f_Y(w).$$

But $f_X(w) = f_Y(w) = \lambda e^{-\lambda w}$, and $F_X(w) = F_Y(w) = 1 - e^{-\lambda w}$, so that

$$f_W(w) = 2\lambda e^{\lambda w} - 2(1 - e^{-\lambda w})\lambda e^{-\lambda w} = 2\lambda e^{-2\lambda w}U(w). \qquad (8\text{-}50)$$

Thus min (X, Y) is also exponential with parameter 2λ.

Example 8.12: Suppose X and Y are as give in the above example. Define $Z = \left[\min(X, Y) / \max(X, Y)\right]$. Determine $f_Z(z)$.

Solution: Although $\min(\cdot)/\max(\cdot)$ represents a complicated function, by partitioning the whole space as before, it is possible to simplify this function. In fact

$$Z = \begin{cases} X/Y, & X \le Y, \\ Y/X, & X > Y. \end{cases} \tag{8-51}$$

As before, this gives

$$\begin{aligned} F_z(z) = P(Z \le z) &= P(X/Y \le z, X \le Y) + P(Y/X \le z, X > Y) \\ &= P(X \le Yz, X \le Y) + P(Y \le Xz, X > Y) . \end{aligned} \tag{8-52}$$

Since X and Y are both positive random variables in this case, we have $0 < z < 1.$ The shaded regions in Figs 8.14 (a)-(b) represent the two terms in the above sum.

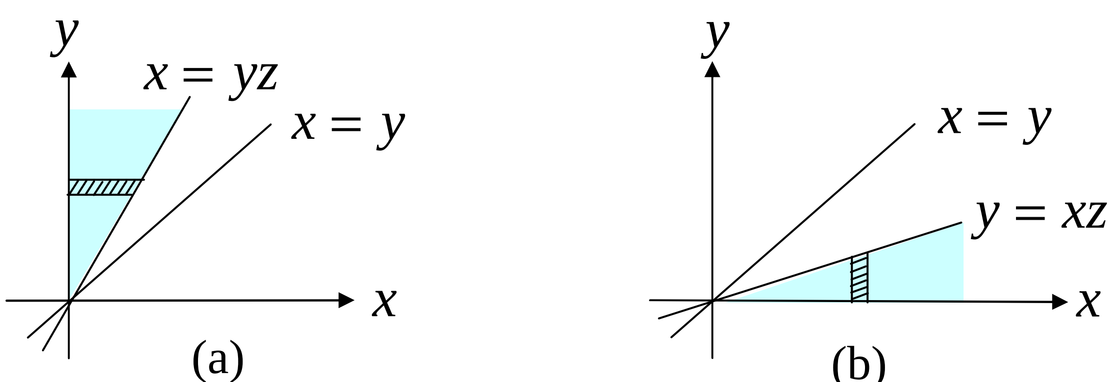


Fig. 8.14

From Fig. 8.14

$$F_Z(z) = \int_0^\infty \int_{x=0}^{yz} f_{XY}(x,y)dxdy + \int_0^\infty \int_{y=0}^{xz} f_{XY}(x,y)dydx. \qquad (8\text{-}53)$$

Hence

$$f_Z(z) = \int_0^\infty y\ f_{XY}(yz,y)dy + \int_0^\infty x\ f_{XY}(x,xz)dx = \int_0^\infty y\{ f_{XY}(yz,y) + f_{XY}(y,yz)\}dy$$

$$= \int_0^\infty y\lambda^2 \{ e^{-\lambda(yz+y)} + e^{-\lambda(y+yz)}\}dy = 2\lambda^2 \int_0^\infty ye^{-\lambda(1+z)y}dy = \frac{2}{(1+z)^2}\int_0^\infty ue^{-u}dy$$

$$= \begin{cases} \dfrac{2}{(1+z)^2}, & 0 < z < 1, \\ 0, & \text{otherwise}. \end{cases} \qquad (8\text{-}54)$$

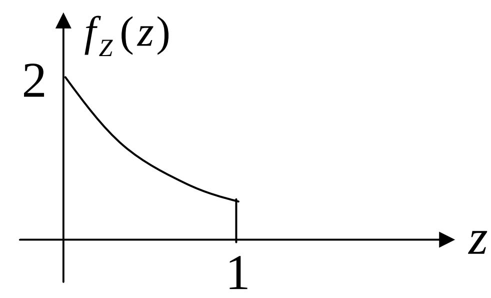


Fig. 8.15

Example 8.13 (Discrete Case): Let X and Y be independent Poisson random variables with parameters λ_1 and λ_2 respectively. Let $Z = X + Y$. Determine the p.m.f of Z.

Solution: Since X and Y both take integer values $\{0, 1, 2, \cdots\}$, the same is true for Z. For any $n = 0, 1, 2, \cdots,$ $X + Y = n$ gives only a finite number of options for X and Y. In fact, if $X = 0$, then Y must be n; if $X = 1$, then Y must be n-1, etc. Thus the event $\{X + Y = n\}$ is the union of $(n + 1)$ mutually exclusive events A_k given by

$$A_k = \{ X = k, \;\; Y = n - k \}, \qquad k = 0, 1, 2, \cdots, n. \qquad (8\text{-}55)$$

As a result

$$P(Z = n) = P(X + Y = n) = P\left(\bigcup_{k=0}^{n} (X = k, \; Y = n - k) \right)$$

$$= \sum_{k=0}^{n} P(X = k, \; Y = n - k). \qquad (8\text{-}56)$$

If X and Y are also independent, then

$$P\left(X = k, \; Y = n - k \right) = P(X = k) P(Y = n - k)$$

and hence

$$P(Z=n)=\sum_{k=0}^{n} P(X=k,\ Y=n-k)$$

$$=\sum_{k=0}^{n} e^{-\lambda_1}\frac{\lambda_1^k}{k!}e^{-\lambda_2}\frac{\lambda_2^{n-k}}{(n-k)!}=\frac{e^{-(\lambda_1+\lambda_2)}}{n!}\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}\lambda_1^k\lambda_2^{n-k}$$

$$=e^{-(\lambda_1+\lambda_2)}\frac{(\lambda_1+\lambda_2)^n}{n!},\qquad n=0,1,2,\cdots,\infty. \tag{8-57}$$

Thus Z represents a Poisson random variable with parameter $\lambda_1+\lambda_2$, indicating that sum of independent Poisson random variables is also a Poisson random variable whose parameter is the sum of the parameters of the original random variables.

As the last example illustrates, the above procedure for determining the p.m.f of functions of discrete random variables is somewhat tedious. As we shall see in Lecture 10, the joint characteristic function can be used in this context to solve problems of this type in an easier fashion.