

9. Two Functions of Two Random Variables

In the spirit of the previous lecture, let us look at an immediate generalization: Suppose X and Y are two random variables with joint p.d.f $f_{XY}(x, y)$. Given two functions $g(x, y)$ and $h(x, y)$, define the new random variables

$$Z = g(X, Y) \quad (9-1)$$

$$W = h(X, Y). \quad (9-2)$$

How does one determine their joint p.d.f $f_{ZW}(z, w)$? Obviously with $f_{ZW}(z, w)$ in hand, the marginal p.d.fs $f_Z(z)$ and $f_W(w)$ can be easily determined.

The procedure is the same as that in (8-3). In fact for given z and w ,

$$\begin{aligned} F_{ZW}(z, w) &= P(Z(\xi) \leq z, W(\xi) \leq w) = P(g(X, Y) \leq z, h(X, Y) \leq w) \\ &= P((X, Y) \in D_{z,w}) = \int \int_{(x,y) \in D_{z,w}} f_{XY}(x, y) dx dy, \end{aligned} \quad (9-3)$$

where $D_{z,w}$ is the region in the xy plane such that the inequalities $g(x, y) \leq z$ and $h(x, y) \leq w$ are simultaneously satisfied.

We illustrate this technique in the next example.

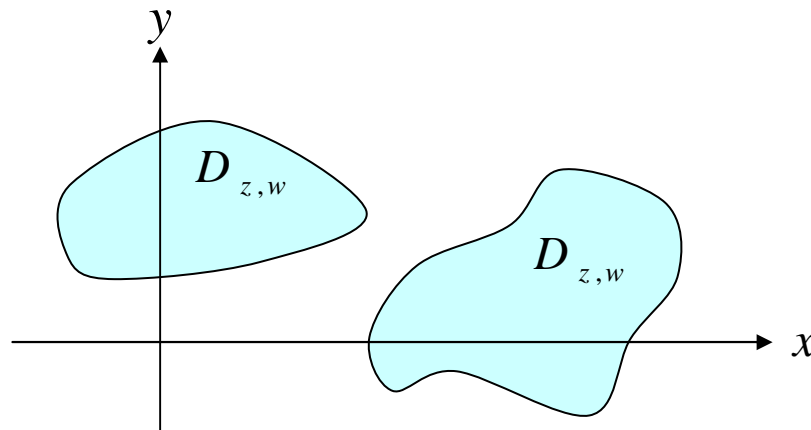


Fig. 9.1

Example 9.1: Suppose X and Y are independent uniformly distributed random variables in the interval $(0, \theta)$.

Define $Z = \min(X, Y)$, $W = \max(X, Y)$. Determine $f_{ZW}(z, w)$.

Solution: Obviously both w and z vary in the interval $(0, \theta)$.

Thus $F_{ZW}(z, w) = 0$, if $z < 0$ or $w < 0$. (9-4)

$$F_{ZW}(z, w) = P(Z \leq z, W \leq w) = P(\min(X, Y) \leq z, \max(X, Y) \leq w). \quad (9-5)$$

We must consider two cases: $w \geq z$ and $w < z$, since they give rise to different regions for $D_{z,w}$ (see Figs. 9.2 (a)-(b)).

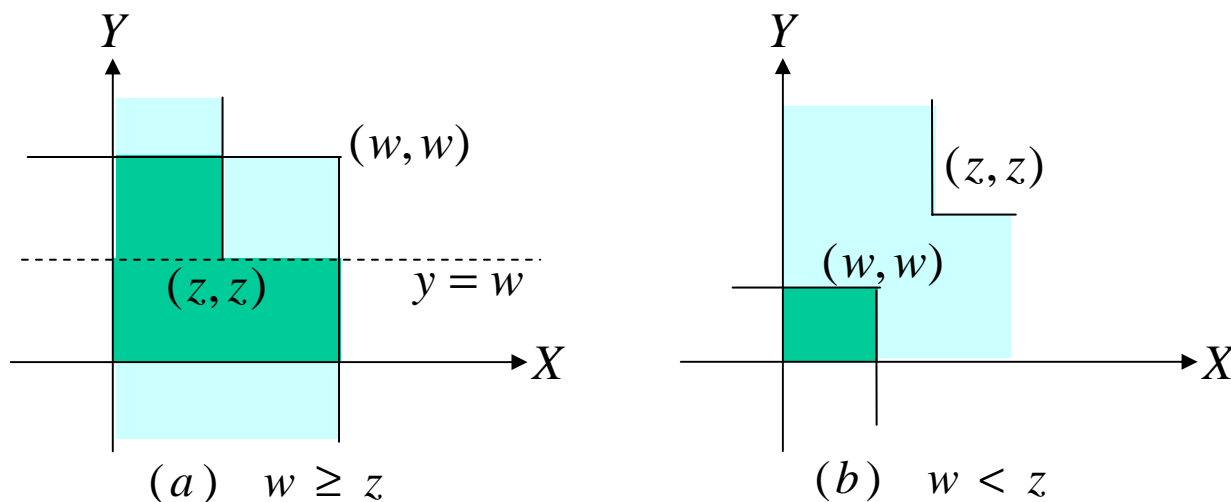


Fig. 9.2

For $w \geq z$, from Fig. 9.2 (a), the region $D_{z,w}$ is represented by the doubly shaded area. Thus

$$F_{ZW}(z, w) = F_{XY}(z, w) + F_{XY}(w, z) - F_{XY}(z, z), \quad w \geq z, \quad (9-6)$$

and for $w < z$, from Fig. 9.2 (b), we obtain

$$F_{ZW}(z, w) = F_{XY}(w, w), \quad w < z. \quad (9-7)$$

With

$$F_{XY}(x, y) = F_X(x) F_Y(y) = \frac{x}{\theta} \cdot \frac{y}{\theta} = \frac{xy}{\theta^2}, \quad (9-8)$$

we obtain

$$F_{ZW}(z, w) = \begin{cases} (2w - z)z / \theta^2, & 0 < z < w < \theta, \\ w^2 / \theta^2, & 0 < w < z < \theta. \end{cases} \quad (9-9)$$

Thus

$$f_{ZW}(z, w) = \begin{cases} 2 / \theta^2, & 0 < z < w < \theta, \\ 0, & \text{otherwise} . \end{cases} \quad (9-10)$$

From (9-10), we also obtain

$$f_Z(z) = \int_z^\theta f_{ZW}(z, w)dw = \frac{2}{\theta} \left(1 - \frac{z}{\theta}\right), \quad 0 < z < \theta, \quad (9-11)$$

and

$$f_W(w) = \int_0^w f_{ZW}(z, w)dz = \frac{2w}{\theta^2}, \quad 0 < w < \theta. \quad (9-12)$$

If $g(x, y)$ and $h(x, y)$ are continuous and differentiable functions, then as in the case of one random variable (see (5-30)) it is possible to develop a formula to obtain the joint p.d.f $f_{ZW}(z, w)$ directly. Towards this, consider the equations

$$g(x, y) = z, \quad h(x, y) = w. \quad (9-13)$$

For a given point (z, w) , equation (9-13) can have many solutions. Let us say

$$(x_1, y_1), (x_2, y_2), \dots, (x_n, y_n),$$

represent these multiple solutions such that (see Fig. 9.3)

$$g(x_i, y_i) = z, \quad h(x_i, y_i) = w. \quad (9-14)$$

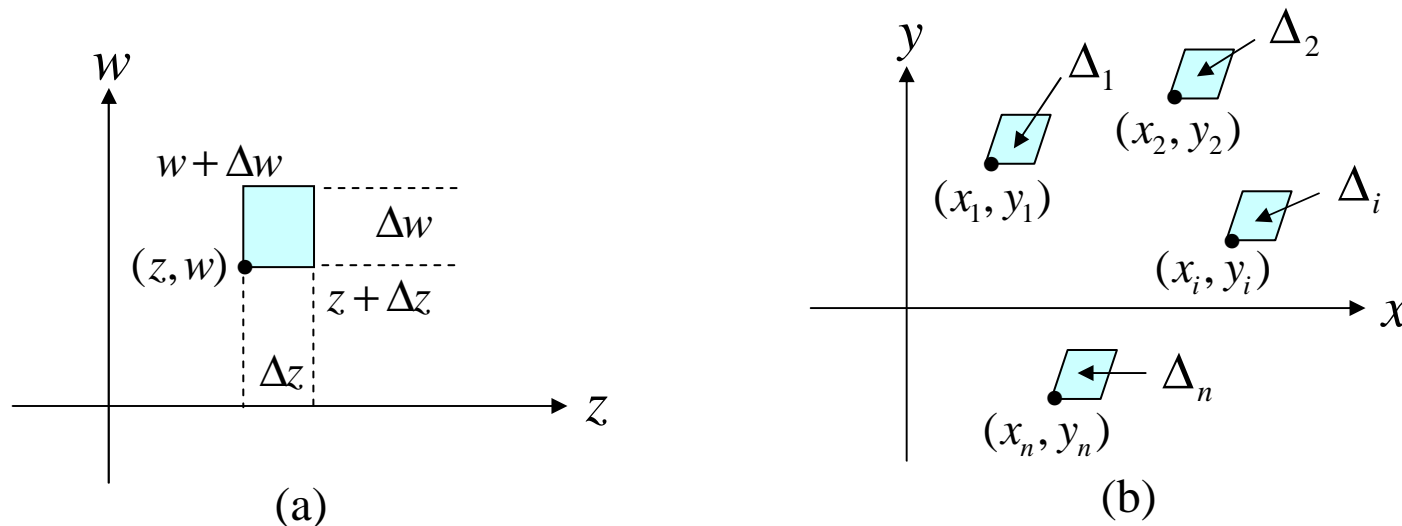


Fig. 9.3

Consider the problem of evaluating the probability

$$\begin{aligned} &P(z < Z \leq z + \Delta z, w < W \leq w + \Delta w) \\ &= P(z < g(X, Y) \leq z + \Delta z, w < h(X, Y) \leq w + \Delta w). \end{aligned} \quad (9-15)$$

Using (7-9) we can rewrite (9-15) as

$$P(z < Z \leq z + \Delta z, w < W \leq w + \Delta w) = f_{zW}(z, w) \Delta z \Delta w. \quad (9-16)$$

But to translate this probability in terms of $f_{XY}(x, y)$, we need to evaluate the equivalent region for $\Delta z \Delta w$ in the xy plane.

Towards this referring to Fig. 9.4, we observe that the point A with coordinates (z, w) gets mapped onto the point A' with coordinates (x_i, y_i) (as well as to other points as in Fig. 9.3(b)).

As z changes to $z + \Delta z$ to point B in Fig. 9.4 (a), let B' represent its image in the xy plane. Similarly as w changes to $w + \Delta w$ to C , let C' represent its image in the xy plane.

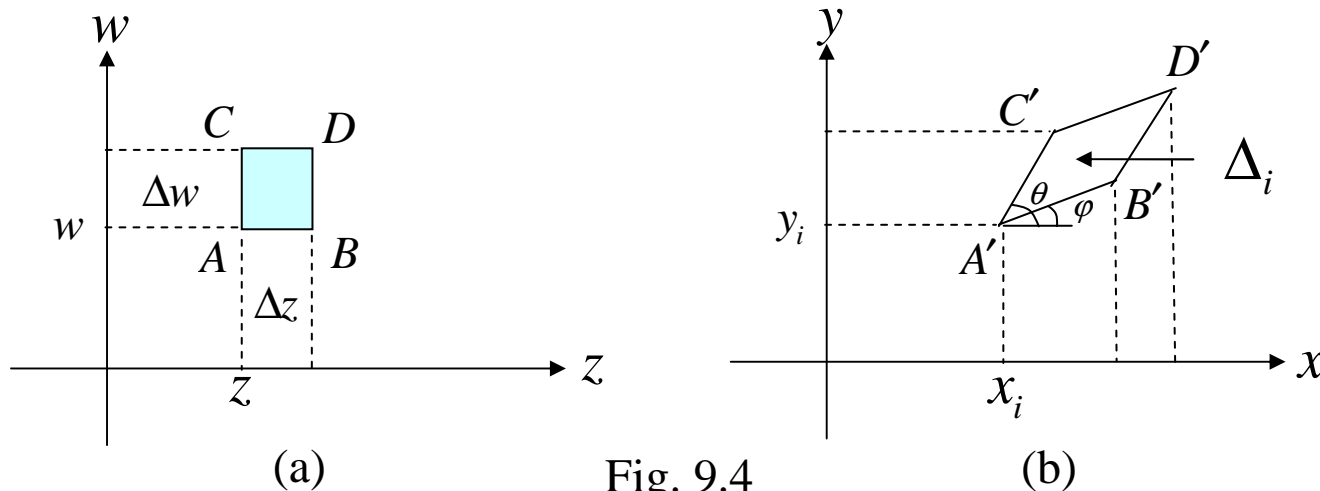


Fig. 9.4

Finally D goes to D' , and $A'B'C'D'$ represents the equivalent parallelogram in the XY plane with area Δ_i . Referring back to Fig. 9.3, the probability in (9-16) can be alternatively expressed as

$$\sum_i P((X, Y) \in \Delta_i) = \sum_i f_{XY}(x_i, y_i) \Delta_i. \quad (9-17)$$

Equating (9-16) and (9-17) we obtain

$$f_{ZW}(z, w) = \sum_i f_{XY}(x_i, y_i) \frac{\Delta_i}{\Delta z \Delta w}. \quad (9-18)$$

To simplify (9-18), we need to evaluate the area Δ_i of the parallelograms in Fig. 9.3 (b) in terms of $\Delta z \Delta w$. Towards this, let g_1 and h_1 denote the inverse transformation in (9-14), so that

$$x_i = g_1(z, w), \quad y_i = h_1(z, w). \quad (9-19)$$

As the point (z, w) goes to $(x_i, y_i) \equiv A'$, the point $(z + \Delta z, w) \rightarrow B'$, the point $(z, w + \Delta w) \rightarrow C'$, and the point $(z + \Delta z, w + \Delta w) \rightarrow D'$.

Hence the respective x and y coordinates of B' are given by

$$g_1(z + \Delta z, w) = g_1(z, w) + \frac{\partial g_1}{\partial z} \Delta z = x_i + \frac{\partial g_1}{\partial z} \Delta z, \quad (9-20)$$

and

$$h_1(z + \Delta z, w) = h_1(z, w) + \frac{\partial h_1}{\partial z} \Delta z = y_i + \frac{\partial h_1}{\partial z} \Delta z. \quad (9-21)$$

Similarly those of C' are given by

$$x_i + \frac{\partial g_1}{\partial w} \Delta w, \quad y_i + \frac{\partial h_1}{\partial w} \Delta w. \quad (9-22)$$

The area of the parallelogram $A'B'C'D'$ in Fig. 9.4 (b) is given by

$$\begin{aligned} \Delta_i &= (A'B')(A'C') \sin(\theta - \varphi) \\ &= (A'B' \cos \varphi)(A'C' \sin \theta) - (A'B' \sin \varphi)(A'C' \cos \theta). \end{aligned} \quad (9-23)$$

But from Fig. 9.4 (b), and (9-20) - (9-22)

$$A'B' \cos \varphi = \frac{\partial g_1}{\partial z} \Delta z, \quad A'C' \sin \theta = \frac{\partial h_1}{\partial w} \Delta w, \quad (9-24)$$

$$A'B' \sin \varphi = \frac{\partial h_1}{\partial z} \Delta z, \quad A'C' \cos \theta = \frac{\partial g_1}{\partial w} \Delta w. \quad (9-25)$$

so that

$$\Delta_i = \left(\frac{\partial g_1}{\partial z} \frac{\partial h_1}{\partial w} - \frac{\partial g_1}{\partial w} \frac{\partial h_1}{\partial z} \right) \Delta z \Delta w \quad (9-26)$$

and

$$\frac{\Delta_i}{\Delta z \Delta w} = \left(\frac{\partial g_1}{\partial z} \frac{\partial h_1}{\partial w} - \frac{\partial g_1}{\partial w} \frac{\partial h_1}{\partial z} \right) = \det \begin{pmatrix} \frac{\partial g_1}{\partial z} & \frac{\partial g_1}{\partial w} \\ \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \end{pmatrix} \quad (9-27)$$

The right side of (9-27) represents the Jacobian $J(z, w)$ of the transformation in (9-19). Thus

$$J(z, w) = \det \begin{pmatrix} \frac{\partial g_1}{\partial z} & \frac{\partial g_1}{\partial w} \\ \frac{\partial h_1}{\partial z} & \frac{\partial h_1}{\partial w} \end{pmatrix}. \quad (9-28)$$

Substituting (9-27) - (9-28) into (9-18), we get

$$f_{ZW}(z, w) = \sum_i |J(z, w)| f_{XY}(x_i, y_i) = \sum_i \frac{1}{|J(x_i, y_i)|} f_{XY}(x_i, y_i), \quad (9-29)$$

since

$$|J(z, w)| = \frac{1}{|J(x_i, y_i)|} \quad (9-30)$$

where $J(x_i, y_i)$ represents the Jacobian of the original transformation in (9-13) given by

$$J(x_i, y_i) = \det \begin{pmatrix} \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \\ \frac{\partial h}{\partial x} & \frac{\partial h}{\partial y} \end{pmatrix}_{x=x_i, y=y_i}. \quad (9-31)$$

Next we shall illustrate the usefulness of the formula in (9-29) through various examples:

Example 9.2: Suppose X and Y are zero mean independent Gaussian r.v.s with common variance σ^2 .

Define $Z = \sqrt{X^2 + Y^2}$, $W = \tan^{-1}(Y/X)$, where $|w| \leq \pi/2$.

Obtain $f_{ZW}(z, w)$.

Solution: Here

$$f_{XY}(x, y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}. \quad (9-32)$$

Since

$$z = g(x, y) = \sqrt{x^2 + y^2}; w = h(x, y) = \tan^{-1}(y/x), \quad |w| \leq \pi/2, \quad (9-33)$$

if (x_1, y_1) is a solution pair so is $(-x_1, -y_1)$. From (9-33)

$$\frac{y}{x} = \tan w, \quad \text{or} \quad y = x \tan w. \quad (9-34)$$

Substituting this into z , we get

$$z = \sqrt{x^2 + y^2} = x\sqrt{1 + \tan^2 w} = x \sec w, \quad \text{or} \quad x = z \cos w. \quad (9-35)$$

and

$$y = x \tan w = z \sin w. \quad (9-36)$$

Thus there are two solution sets

$$x_1 = z \cos w, \quad y_1 = z \sin w, \quad x_2 = -z \cos w, \quad y_2 = -z \sin w. \quad (9-37)$$

We can use (9-35) - (9-37) to obtain $J(z, w)$. From (9-28)

$$J(z, w) = \begin{vmatrix} \frac{\partial x}{\partial z} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial z} & \frac{\partial y}{\partial w} \end{vmatrix} = \begin{vmatrix} \cos w & -z \sin w \\ \sin w & z \cos w \end{vmatrix} = z, \quad (9-38)$$

so that

$$|J(z, w)| = z. \quad (9-39)$$

We can also compute $J(x, y)$ using (9-31). From (9-33),

$$J(x, y) = \begin{vmatrix} \frac{x}{\sqrt{x^2 + y^2}} & \frac{y}{\sqrt{x^2 + y^2}} \\ \frac{-y}{x^2 + y^2} & \frac{x}{x^2 + y^2} \end{vmatrix} = \frac{1}{\sqrt{x^2 + y^2}} = \frac{1}{z}. \quad (9-40)$$

Notice that $|J(z, w)| = 1/|J(x_i, y_i)|$, agreeing with (9-30).

Substituting (9-37) and (9-39) or (9-40) into (9-29), we get

$$\begin{aligned} f_{ZW}(z, w) &= z(f_{XY}(x_1, y_1) + f_{XY}(x_2, y_2)) \\ &= \frac{z}{\pi\sigma^2} e^{-z^2/2\sigma^2}, \quad 0 < z < \infty, \quad |w| < \frac{\pi}{2}. \end{aligned} \quad (9-41)$$

Thus

$$f_Z(z) = \int_{-\pi/2}^{\pi/2} f_{ZW}(z, w) dw = \frac{z}{\sigma^2} e^{-z^2/2\sigma^2}, \quad 0 < z < \infty, \quad (9-42)$$

which represents a Rayleigh r.v with parameter σ^2 , and

$$f_W(w) = \int_0^\infty f_{ZW}(z, w) dz = \frac{1}{\pi}, \quad |w| < \frac{\pi}{2}, \quad (9-43)$$

which represents a uniform r.v in the interval $(-\pi/2, \pi/2)$.
 Moreover by direct computation

$$f_{ZW}(z, w) = f_Z(z) \cdot f_W(w) \quad (9-44)$$

implying that Z and W are independent. We summarize these results in the following statement: If X and Y are zero mean independent Gaussian random variables with common variance, then $\sqrt{X^2 + Y^2}$ has a Rayleigh distribution and $\tan^{-1}(Y/X)$ has a uniform distribution. Moreover these two derived r.vs are statistically independent. Alternatively, with X and Y as independent zero mean r.vs as in (9-32), $X + jY$ represents a complex Gaussian r.v. But

$$X + jY = Ze^{jW}, \quad (9-45)$$

where Z and W are as in (9-33), except that for (9-45) to hold good on the entire complex plane we must have $-\pi < W < \pi$, and hence it follows that the magnitude and phase of

a complex Gaussian r.v are independent with Rayleigh and uniform distributions ($U \sim (-\pi, \pi)$) respectively. The statistical independence of these derived r.vs is an interesting observation.

Example 9.3: Let X and Y be independent exponential random variables with common parameter λ . Define $U = X + Y$, $V = X - Y$. Find the joint and marginal p.d.f of U and V .

Solution: It is given that

$$f_{XY}(x, y) = \frac{1}{\lambda^2} e^{-(x+y)/\lambda}, \quad x > 0, \quad y > 0. \quad (9-46)$$

Now since $u = x + y$, $v = x - y$, always $|v| < u$, and there is only one solution given by

$$x = \frac{u + v}{2}, \quad y = \frac{u - v}{2}. \quad (9-47)$$

Moreover the Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -2$$

and hence

$$f_{UV}(u, v) = \frac{1}{2\lambda^2} e^{-u/\lambda}, \quad 0 < |v| < u < \infty, \quad (9-48)$$

represents the joint p.d.f of U and V . This gives

$$f_U(u) = \int_{-u}^u f_{UV}(u, v) dv = \frac{1}{2\lambda^2} \int_{-u}^u e^{-u/\lambda} dv = \frac{u}{\lambda^2} e^{-u/\lambda}, \quad 0 < u < \infty, \quad (9-49)$$

and

$$f_V(v) = \int_{|v|}^{\infty} f_{UV}(u, v) du = \frac{1}{2\lambda^2} \int_{|v|}^{\infty} e^{-u/\lambda} du = \frac{1}{2\lambda} e^{-|v|/\lambda}, \quad -\infty < v < \infty. \quad (9-50)$$

Notice that in this case the r.vs U and V are not independent.

As we show below, the general transformation formula in (9-29) making use of two functions can be made useful even when only one function is specified.

Auxiliary Variables:

Suppose

$$Z = g(X, Y), \quad (9-51)$$

where X and Y are two random variables. To determine $f_Z(z)$ by making use of the above formulation in (9-29), we can define an auxiliary variable

$$W = X \quad \text{or} \quad W = Y \quad (9-52)$$

and the p.d.f of Z can be obtained from $f_{ZW}(z, w)$ by proper integration.

Example 9.4: Suppose $Z = X + Y$ and let $W = Y$ so that the transformation is one-to-one and the solution is given by $y_1 = w, \quad x_1 = z - w$.

The Jacobian of the transformation is given by

$$J(x, y) = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = 1$$

and hence

$$f_{ZW}(x, y) = f_{XY}(x_1, y_1) = f_{XY}(z - w, w)$$

or

$$f_Z(z) = \int f_{ZW}(z, w)dw = \int_{-\infty}^{+\infty} f_{XY}(z - w, w)dw, \quad (9-53)$$

which agrees with (8.7). Note that (9-53) reduces to the convolution of $f_X(z)$ and $f_Y(z)$ if X and Y are independent random variables. Next, we consider a less trivial example.

Example 9.5: Let $X \sim U(0,1)$ and $Y \sim U(0,1)$ be independent. Define $Z = (-2 \ln X)^{1/2} \cos(2\pi Y)$. (9-54)

Find the density function of Z .

Solution: We can make use of the auxiliary variable $W = Y$ in this case. This gives the only solution to be

$$x_1 = e^{-(z \sec(2\pi w))^2 / 2}, \quad (9-55)$$

$$y_1 = w, \quad (9-56)$$

and using (9-28)

$$J(z, w) = \begin{vmatrix} \frac{\partial x_1}{\partial z} & \frac{\partial x_1}{\partial w} \\ \frac{\partial y_1}{\partial z} & \frac{\partial y_1}{\partial w} \end{vmatrix} = \begin{vmatrix} -z \sec^2(2\pi w) e^{-(z \sec(2\pi w))^2 / 2} & \frac{\partial x_1}{\partial w} \\ 0 & 1 \end{vmatrix} \quad (9-57)$$

$$= -z \sec^2(2\pi w) e^{-(z \sec(2\pi w))^2 / 2}.$$

Substituting (9-55) - (9-57) into (9-29), we obtain

$$f_{ZW}(z, w) = |z| \sec^2(2\pi w) e^{-(z \sec(2\pi w))^2 / 2}, \quad (9-58)$$

$$-\infty < z < +\infty, \quad 0 < w < 1,$$

and

$$f_Z(z) = \int_0^1 f_{ZW}(z, w)dw = e^{-z^2/2} \int_0^1 |z| \sec^2(2\pi w) e^{-(|z| \tan(2\pi w))^2/2} dw. \quad (9-59)$$

Let $u = |z| \tan(2\pi w)$ so that $du = 2\pi |z| \sec^2(2\pi w)dw$. Notice that as w varies from 0 to 1, u varies from $-\infty$ to $+\infty$.

Using this in (9-59), we get

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \underbrace{\int_{-\infty}^{+\infty} e^{-u^2/2} \frac{du}{\sqrt{2\pi}}}_1 = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}, \quad -\infty < z < \infty, \quad (9-60)$$

which represents a zero mean Gaussian r.v with unit variance. Thus $Z \sim N(0,1)$. Equation (9-54) can be used as a practical procedure to generate Gaussian random variables from two independent uniformly distributed random sequences.