California Institute of Technology
Department of Electrical Engineering

EE 162
Random Processes for Communication and Signal Processing

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5. *The Random Radio*

In 1883, when interest was at a fever pitch for finding a way to confirm the prediction by the Scottish mathematical physicist James Clerk Maxwell (1831-1879) of the existence of electromagnetic waves traveling through empty space at the speed of light, the Irish physicist George Francis FitzGerald (1851-1901) made a suggestion. In one of the shortest scientific papers ever published (a mere one paragraph), he described a charged capacitor being allowed to suddenly discharge through an inductive circuit (one with lots of wire wrapped around an iron core, for example). If the component values are properly chosen, then the discharge is in the form of a high-frequency oscillation. Perhaps, suggested FitzGerald, these oscillations would be sufficient to launch electromagnetic energy into space and thus create Maxwell's waves. Four years later, the German physicist Heinrich Hertz (1857-1894) did just that, and then others turned the physics into what soon became the worldwide sensation of commercial radio.

Without going through the details of FitzGerald's math, what his work boiled down to is essentially the following: Given an inductor (e.g., a coil of wire), a capacitor (a component that stores electricity and that, even in FitzGerald's day, could be easily constructed from nothing more than tinfoil and a glass jar), and the inherent resistance of the circuit itself, it is not hard to show that the discharge current decays exponentially with time. The way the exponential decay occurs can be either monotonic or oscillatory, however, and which decay mode occurs depends on the component values. It turns out that what determines the mode is a particular quadratic equation, with coefficients determined by the component values. If the quadratic has real solutions then one mode occurs, and if the solutions are complex then the other mode occurs.

So, imagine that an early radio experimenter had several boxes of components on his workbench: one containing an assortment of capacitors, another containing various inductors, and a third with various resistors. If he simply grabbed components at random and connected them into FitzGerald's circuit, then the resulting current might or might not oscillate. This simple illustration leads us to the following pure mathematics question in probability.

Consider the partially random quadratic equation $x^2 + Bx + C = 0$, where B and C are independent random variables uniform from 0 to 1 (I call this partially random since the coefficient of x^2 is not random). What is the probability that the solution to the partially random quadratic is real? This is a classic problem in geometric probability. It is a straightforward question to answer. From the quadratic formula, we have

$$x = \frac{-B \pm \sqrt{B^2 - 4C}}{2},$$

which tells us that x is real if and only if $B^2 \geq 4C$. That is, we wish to calculate the probability that $C \leq \frac{1}{4}B^2$. The sample space for B and C, on which this inequality is sketched, looks like the figure below, where the shaded region represents the collection of all pairs of values for B and C that result in real roots.

So, the answer to the question is the probability of the shaded region. Since B and C are each uniform and independent, then the probability we want is simply

$$\frac{\text{area of shaded region}}{\text{area of sample space}(=1)} = \int_0^1 \frac{1}{4} B^2 \, dB = \frac{1}{12}$$

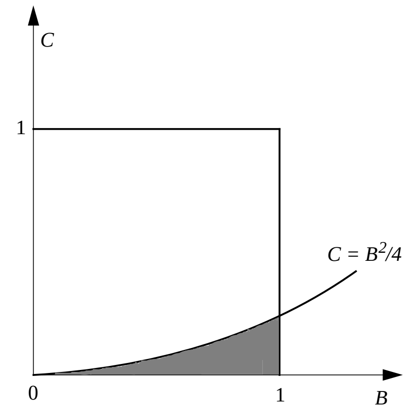

Figure 1: Real roots of quadratic equation.

This problem is also a simple example of the kind of probability question that is very easy to answer by computer simulation. So, using the random number generator available in Matlab, I asked for 100,000 pairs of values for B and C. Checking each pair to see if it satisfied $B^2 \geq 4C$, I got the answer 0.08411, which compares fairly well with $1/12 = 0.08333\ldots$.

A surprisingly simple twist to the partially random quadratic demands a more sophisticated analysis (although it remains elementary). So, consider now the totally random quadratic equation $Ax^2 + Bx + C = 0$, where A is also uniform from 0 to 1 (and independent of both B and C). What is the probability that the solution to the totally random quadratic is real? You cannot simply divide through by A and declare $x^2 + (B/A)x + (C/A) = 0$ to be a partially random quadratic with the answer computed above. First of all, B/A and C/A aren't uniform; and, in fact, they aren't even independent. This assertion usually astonishes many who argue that since B and C are independent, then why would dividing both by the same value (A) suddenly make the ratios dependent? It does, and you are about to prove it. Here is your assignment:

(a) Calculate the probability of real roots for the totally random quadratic without making any independence assumptions other than that A, B, and C are all independent.

(b) Redo the problem with the assumption that B/A and C/A are independent and show that the resulting answer does not agree with your first calculation. Thus, conclude that the assumption of the independence of B/A and C/A must be wrong.

(c) Write a Matlab simulation for the totally random quadratic and compare the result with your two theoretical calculations.